METHODS OF APPLIED MATHEMATICS COMPREHENSIVE EXAMINATION AUGUST 2017

Work on as many of the following problems as possible. Turn in *all* your work.

(1) Consider the projectile problem of a body (mass m), dropped vertically from a point above the Earth's equator (assume a constant gravitational field and that the body falls on a line connecting its drop point to the Earth center). Assuming that the object always experiences a cubic drag force as a function of its velocity:

(a) Write down the Newtonian dynamics.

(b) Non-dimensionalize with respect to viscous scales (terminal velocity and initial height), identify non-dimensional parameters.

(c) Evaluate a perturbation solution in the small non-dimensional parameter and sketch the solution

(d) For an observer on the Earth's equator, write the leading order trajectory from the observer's perspective (assuming the Earth is in constant rotation with respect to the north-south pole axis).

(2) Consider the eigenvalue problem on the real line $x \in \mathbb{R}$

$$\epsilon y'' - (U(x) + \lambda)y = 0, \qquad y(x) \to 0, \text{ as } |x| \to \infty.$$

with the potential:

$$U(x) = a\,\delta(x - L) + b\,\delta(x + L)$$

where $\delta(x)$ is the Dirac delta function. First, set $b = 0$ and solve for the eigenvalue(s) and eigenfunction(s), as the parameter a varies over the real line. Second, calculate the eigenvalue(s) and eigenfunctions with when both a and b vary over the real line. How does the single delta function result compare with the double delta case as $L \to \infty$?

(3) Consider the boundary value problem

$$\epsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \epsilon u^2 ,$$

$$u(0, y) = g(y), \quad u(1, y) = 0, \quad u(x, 0) = f(x), \quad u(x, 1) = 0 ,$$

in $(x, y) \in [0, 1] \times [0, 1]$, for some functions $f(x)$, $g(y)$. Find the first term in an asymptotic expansion of the solution as $\epsilon \to 0$. The functions f and g can be taken to be smooth but are otherwise generic (i.e., do not take special values in $[0, 1]$). If you cannot solve any of the differential equations necessary for your procedure, just set them up properly (i.e., state the proper equation with proper initial and/or boundary conditions etc.).

(4) Consider the functions

$$f(z) = \sqrt{\cosh \sqrt{z}}\,, \qquad g(z) = \sqrt{(z - i)(z^2 - 1)}\,, \qquad h(z) = \sqrt{z^4 - 1}$$

of the complex variable z.

(a) Choose branch cuts for f to be single valued in the complex z plane.

(b) Can the function g be defined to be single valued in the domain exterior to the circle $|z| = 2$? Discuss.

(c) Answer part (b) for the function h, and explain.

(5) Find the first term in the asymptotic expansion of

$$f(x) = \int_C \frac{1}{\zeta^3} \exp\left[\mathrm{i}\, x \left(\zeta^2 - \frac{1}{\zeta^2} \right) \right] d\zeta$$

as $x \to +\infty$, where C is a closed contour around the origin.

(6) Consider the rapidly varying diffusivity

$$K(x, \epsilon) = A + G(x/\epsilon)$$

where A is chosen to guarantee K is positive, and ϵ is a small constant. Also consider the function, $v(x)$. By applying homogenization, average the following advection-diffusion equation

$$\frac{\partial u}{\partial t} + \frac{1}{\epsilon} v(x/\epsilon) \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(K(x,\epsilon) \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y}\left(K(x,\epsilon) \frac{\partial u}{\partial y} \right)$$

$$u(x, y, 0) = u_0(x, y)\,,$$

and calculate a leading order effective equation governing the evolution as $\epsilon \to 0$ over the (x, y)-plane, assuming the functions $G(x/\epsilon)$, $v(x/\epsilon)$ are mean zero, periodic, and share the same period. Solve the averaged equation in free space. If instead we had $v(x/\epsilon^2)$, how would things change?

(7) (a) Find two term asymptotic expansions as $\epsilon \to 0$ for roots of the equation

$$\epsilon z^7 + (z - 5)^2 = \epsilon z$$

(analyze carefully the unperturbed root at $z = 5$, and analyze one root coming in from infinity.)

(b) Find a two term asymptotic expansion for the $x = 0$ unperturbed root of the equation

$$x^2 e^{-x^4} = \epsilon$$

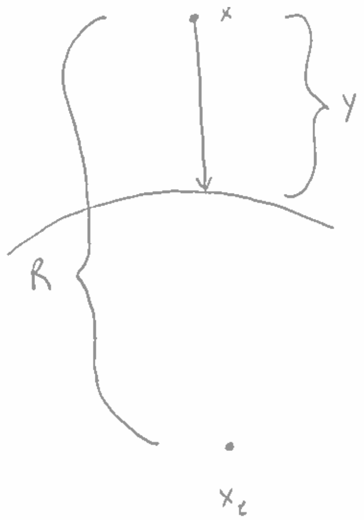
(8) (a) Explain the difference between pointwise convergence and asymptotic convergence. Illustrate with the particular example of power series.

(b) Given a real integral, $\int_a^b f(x, y)dy$, with $f(x, y) \sim f_0(y)$ as $x \to x_0$, under what conditions is termwise integration guaranteed to yield valid asymptotics?

Problem 1

Aug 2017

a)



$$m\ddot{x} = -\frac{GmM}{|x-x_e|^2} - \beta\dot{x}^3$$

$$M\ddot{x}_e = \frac{GmM}{|x-x_e|^2}$$

since $M \gg m$, $\dot{x}_e \ll \dot{x}$ b/c the velocity of an object is inversely proportional to its mass. Thus $\dot{x}_e \approx$ const so $\ddot{x}_e \approx 0$.

$$\begin{cases} \ddot{x} = -\dfrac{GM}{|x-x_e|^2} - \dfrac{\beta}{m}\dot{x}^3 \\ \dot{x}(0) = 0 \\ x(0) = y_0 + R_e \qquad y_0 > 0 \end{cases}$$

change of var:

$$\begin{cases} y = x - R_e \\ x_e = 0 \end{cases} \Rightarrow x = y + \underset{\text{radius of earth}}{R_e}$$

$$\Rightarrow \quad \ddot{y} = -\frac{GM}{|y+R_e|^2} - \frac{\beta}{m}\dot{y}^3$$

$$\boxed{\ddot{y} = -\underbrace{\frac{GM}{R_e^2}}_{g}\frac{1}{\left(1+\frac{y}{R_e}\right)^2} - \frac{\beta}{m}\dot{y}^3}$$

where $y(0) = x(0) - R_e = y_0 > 0$

$\dot{y}(0) = \dot{x}(0) = 0$

b) Let v_t = terminal velocity and $L = \frac{v_t^2}{g}$, $T = \frac{v_t}{g}$.

Choose $y = Lz$ and $t = T\tau \Rightarrow \frac{dt}{d\tau} = T$

non-dim

$$\frac{dy}{dt} = \frac{d(Lz)}{d\tau}\frac{d\tau}{dt} = \frac{L}{T}\frac{dz}{d\tau} = v_t \dot{z}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{L}{T}\frac{dz}{d\tau}\right) = \frac{L}{T^2}\frac{d^2z}{d\tau^2} = g\ddot{z}$$

$$\Rightarrow \quad g\ddot{z} = -g\frac{1}{\left|1+\frac{zL}{R_e}\right|^2} - \frac{\beta}{m}(v_t\dot{z})^3$$

Let $\varepsilon = \frac{L}{R_e}$ and $\alpha = \frac{\beta v_t R_e}{m}$

$$\boxed{\ddot{z} = -\frac{1}{|1+\varepsilon z|^2} - \varepsilon\alpha\dot{z}^3} \qquad (1)$$

$$0 = \dot{y}(0) = v_t\dot{z}(0)$$

$$\dot{z}(0) = 0$$

$$y_0 = y(0) = L\,z(0)$$

$$z(0) = y_0/L = \bar{z}$$

c) We know $v_t^2 << R_e \cdot g$ so $\varepsilon << 1$. Now consider

$$z \sim z_0 + \varepsilon z_1 + \cdots$$

and plug into (1):

$$\ddot{z}_0 + \varepsilon \ddot{z}_1 = -\frac{1}{(1 + \varepsilon(z_0 + \varepsilon z_1 + \ldots))} - \varepsilon \alpha (\dot{z}_0 + \varepsilon \dot{z}_1 + \ldots)^3$$

$$= -1 + 2\varepsilon(z_0 + \varepsilon z_1 + \ldots) + \Theta(\varepsilon^2) - \varepsilon \alpha (\dot{z}_0 + \varepsilon \dot{z}_1 + \ldots)^3$$

$\Theta(1)$: $\ddot{z}_0 = -1$

$$\dot{z}_0 = -\tau + \cancel{a}\to 0$$

$$z_0 = -\frac{\tau^2}{2} + b = -\frac{1}{2}\tau^2 + Z$$

$\Theta(\varepsilon)$: $\ddot{z}_1 = 2z_0 - \alpha \dot{z}_0 = 2(-\tfrac{1}{2}\tau^2 + Z) + \alpha\tau = -\tau^2 + \alpha\tau + Z$

$$\dot{z}_1 = -\frac{1}{3}\tau^3 + \frac{\alpha}{2}\tau^2 + Z\tau + \cancel{a}\to 0$$

$$z_1 = -\frac{1}{12}\tau^4 + \frac{\alpha}{6}\tau^3 + \frac{Z}{2}\tau^2 + \cancel{b}\to 0$$

$$z \sim -\frac{1}{2}\tau^2 + Z + \varepsilon\left(-\frac{1}{12}\tau^4 + \frac{\alpha}{6}\tau^3 + \frac{Z}{2}\tau^2\right)$$

Problem 2 Aug 2017

$$\varepsilon y'' - (U(x) + \lambda)\, y = 0 \qquad \Rightarrow \qquad y'' = \frac{\lambda + U(x)}{\varepsilon}\, y$$

$$U(x) = a\,\delta(x-L) + b\,\delta(x+L)$$

1) $b = 0$

Since we want $y \to 0$ as $|x| \to \infty$, we know $\frac{\lambda + U(x)}{\varepsilon} > 0$ in regions I and II. So $\frac{\lambda}{\varepsilon} > 0$.

Region I & II $\qquad y_i'' = \underbrace{\frac{\lambda}{\varepsilon}}_{>0} y_i \qquad i = 1, 2$

$$y_i = A_i e^{\sqrt{\lambda/\varepsilon}\, x} + B_i e^{-\sqrt{\lambda/\varepsilon}\, x}$$

Since $y \to 0$ as $|x| \to \infty$, we know $B_1 = 0$ and $A_2 = 0$

$$\Rightarrow \quad y_1 = A e^{\sqrt{\lambda/\varepsilon}\, x}$$

$$y_2 = B e^{-\sqrt{\lambda/\varepsilon}\, x}$$

Patching:

$$y_1(L) = y_2(L)$$

$$A e^{L\sqrt{\lambda/\varepsilon}} = B e^{-L\sqrt{\lambda/\varepsilon}}$$

$$B = A e^{2L\sqrt{\lambda/\varepsilon}}$$

Let $0 < \gamma \ll 1$

$$\int_{L-\gamma}^{L+\gamma} y'' = \int_{L-\gamma}^{L+\gamma} \frac{\lambda + U(x)}{\varepsilon} y = \underbrace{\cancel{\int_{L-\gamma}^{L+\gamma} \frac{\lambda}{\varepsilon} y}}_{\to 0 \text{ as } \gamma \to 0} + \int_{L-\gamma}^{L+\gamma} \frac{a}{\varepsilon} y \, \delta(x-L)$$

$$y' \Big|_{L-\gamma}^{L+\gamma} = \frac{a}{\varepsilon} y(L)$$

as $\gamma \to 0$: $\quad y_2'(L) - y_1'(L) = \frac{a}{\varepsilon} y(L)$

$$-B\sqrt{\lambda/\varepsilon}\, e^{-L\sqrt{\lambda/\varepsilon}} - A\sqrt{\lambda/\varepsilon}\, e^{L\sqrt{\lambda/\varepsilon}} = \frac{a}{\varepsilon} A e^{L\sqrt{\lambda/\varepsilon}}$$

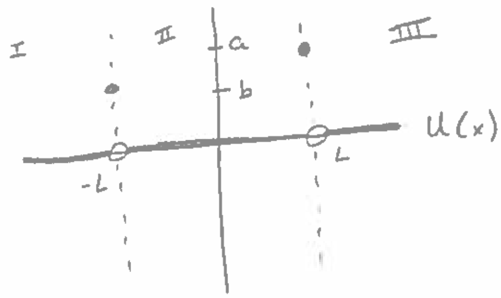
$$-A\sqrt{\lambda/\varepsilon}\, \cancel{e^{\sqrt{\lambda/\varepsilon}}} - \cancel{A}\sqrt{\lambda/\varepsilon}\, \cancel{e^{L\sqrt{\lambda/\varepsilon}}} = \frac{a}{\varepsilon} \cancel{A e^{L\sqrt{\lambda/\varepsilon}}}$$

$$2\sqrt{\lambda/\varepsilon} = -\frac{a}{\varepsilon}$$

$$\frac{\lambda}{\varepsilon} = \frac{a^2}{4\varepsilon^2}$$

$$\boxed{\lambda = \frac{a^2}{4\varepsilon}}$$

2) $b \neq 0$



Again, $y \to 0$ as $|x| \to \infty$ so $\frac{\lambda}{\varepsilon} > 0$.

<u>All Regions:</u> $i = 1, 2, 3$

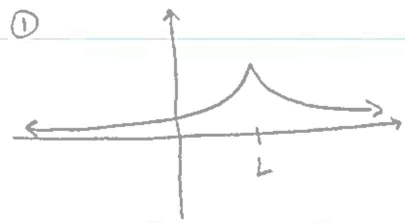
$$y_i = A_i e^{\sqrt{\lambda/\varepsilon}\, x} + B_i e^{-\sqrt{\lambda/\varepsilon}\, x}$$

where $A_3 = 0$ and $B_1 = 0$ to ensure $y \to 0$ as $|x| \to \infty$

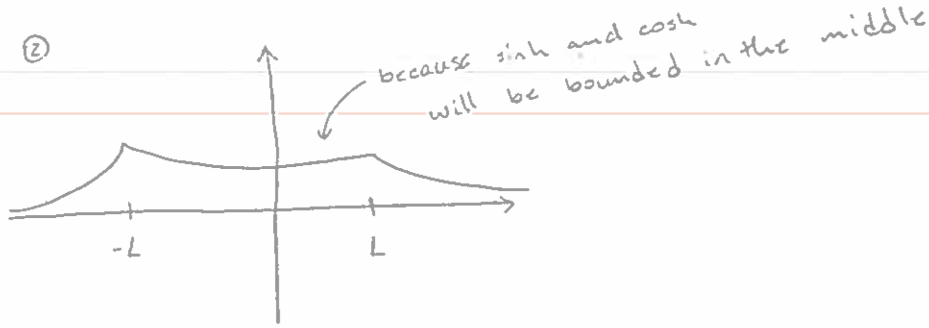
$$\Rightarrow \begin{cases} y_1 = A_1 e^{\sqrt{\lambda/\varepsilon}\, x} \\ y_2 = A_2 e^{\sqrt{\lambda/\varepsilon}\, x} + B_2 e^{-\sqrt{\lambda/\varepsilon}\, x} \\ y_3 = B_3 e^{-\sqrt{\lambda/\varepsilon}\, x} \end{cases}$$

To find eigenvalues, one must proceed with a patching process simillar to the one in part 1.

The result from part 1 looks like:



While the result in part 2 looks like:

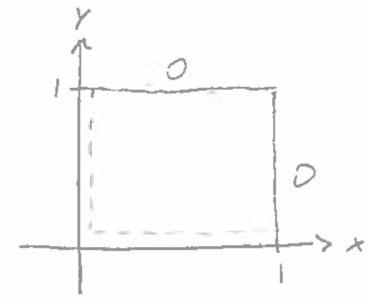


If $L \to \infty$, you can see soln ① will stretch out the left portion and essentially y will be zero. But soln ② will have the middle section stretched out and its will more closely represent the constant line $y = \frac{a+b}{2}$.

Problem 3

Aug 2017

$$\begin{cases} \varepsilon u_{xx} + u_{yy} = \varepsilon u^2 \\ u(0,y) = g(y) \quad , \quad u(x,0) = f(x) \quad \text{outter BC's} \\ u(1,y) = 0 \quad , \quad u(x,1) = 0 \quad \text{inner} \end{cases}$$



$\to O(1)$

Outter: $u_{yy} = 0$

$$u(x,y) = a(x)\,y + b(x)$$

$$u(0,y) = a(0)\,y + b(0) = g(y)$$

$$u(x,0) = b(x) = f(x)$$

Inner:

$x=1$: $\quad z = \dfrac{x-1}{\varepsilon} \quad \Rightarrow \quad \dfrac{d}{dx} = \dfrac{1}{\varepsilon}\dfrac{d}{dz}$

$$\frac{1}{\varepsilon} v_{xx} + v_{yy} = \varepsilon v^2$$

$y=1$: $\quad z = \dfrac{y-1}{\varepsilon} \quad \Rightarrow \quad \dfrac{d}{dy} = \dfrac{1}{\varepsilon}\dfrac{d}{dz}$

Problem 4

a) $f(z) = \sqrt{\cosh(\sqrt{z})}$

Branch points:

1) $z = 0$

2) $\cosh(\sqrt{z}) = 0$

$\cos(i\sqrt{z}) = 0$

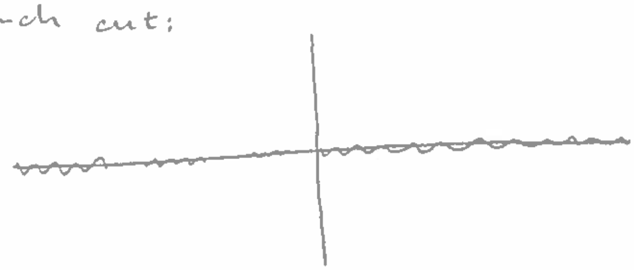
$\cos(\sqrt{-z}) = 0$

$\sqrt{-z} = \frac{\pi}{2} + \pi n$

$-z = (\frac{\pi}{2} + \pi n)^2$

$z = -(\frac{\pi}{2} + \pi n)^2 = -\frac{\pi^2}{4}, -\frac{9\pi^2}{4}, -\frac{25\pi^2}{4}$

Branch cut:



b) $g(z) = \sqrt{(z-i)(z^2-1)}$

Branch points: $z = i, \pm 1$

One of the branch cuts will have to go to infinity to ensure $g(z)$ is single valued.

By excluding $|z| \leq 2$, you lose the branch points you'd need to make the branch cut so $g(z)$ can't be s.v. on this domain.

c) $h(z) = \sqrt{z^4 - 1}$

Branch points: $z = \pm i, \pm 1$

All branch cuts are made on the interior of $|z| \le 2$ to make $h(z)$ s.v. so removing $|z| \le 2$ from the $h(z)$ will not cause any problems.

Problem 5

Aug 2017

$$f(x) = \int_C \frac{1}{s^3} e^{ix(s^2 - \frac{1}{s^2})} ds$$

No branch cuts, just a 3^{rd} order pole at $s=0$.

Let $s = e^{i\theta} \Rightarrow ds = -ie^{i\theta} d\theta$ where $\theta \in [0, 2\pi)$

$$f(x) = i\int_0^{2\pi} \frac{1}{e^{3i\theta}} e^{ix(e^{2\theta i} - e^{-2\theta i})} e^{i\theta} d\theta \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$= i\int_0^{2\pi} e^{-2\theta i} e^{-2x\sin(2\theta)} d\theta$$

$$= i\int_0^{2\pi} \cos(2\theta) e^{-2x\sin(2\theta)} d\theta + \int_0^{2\pi} \sin(2\theta) e^{-2x\sin(2\theta)} d\theta$$

Now the function is defined in $\mathbb{R}$ and we are integrating in $\mathbb{R}$ so we can use Laplace's Method.

$g_1(\theta) = \cos(2\theta) \qquad g_2(\theta) = \sin(2\theta)$

$h(\theta) = -2\sin(2\theta)$

$h'(\theta) = -4\cos(2\theta) = 0 \qquad \theta^* = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$$h(\tfrac{\pi}{4}) = h(\tfrac{5\pi}{4}) = -2\sin(\tfrac{\pi}{2}) = -2$$

dominant $\longrightarrow$ $\underset{\theta_1}{h(\tfrac{3\pi}{4})} = \underset{\theta_2}{h(\tfrac{7\pi}{4})} = -2\sin(\tfrac{3\pi}{2}) = 2$

θ_1 and θ_2 have equal weights

$$h''(\theta) = 8\sin(2\theta)$$

$$h''(\theta_i) = 8\sin\left(\frac{3\pi}{2}\right) = -8$$

$$f(x) \sim \sum_{j=1}^{2} \left(i g_1(\theta_j) + g_2(\theta_j)\right) e^{x h(\theta_1)} \sqrt{\frac{2\pi}{x|h''(\theta_1)|}} \quad \text{as} \quad x \to \infty$$

$$g_1(\theta_1) = \cos\left(\frac{3\pi}{2}\right) = 0 \qquad g_2(\theta_1) = \sin\left(\frac{3\pi}{2}\right) = -1$$

$$g_1(\theta_2) = 0 \qquad g_2(\theta_2) = -1$$

$$f(x) \sim -2e^{2x}\sqrt{\frac{\pi}{4x}} \quad \text{as} \quad x \to \infty$$

$$\boxed{f(x) = -e^{2x}\sqrt{\frac{\pi}{x}} \quad \text{as} \quad x \to \infty}$$

<u>Problem 6</u> Aug 2017

$$\begin{cases} u_t + \frac{1}{\varepsilon} v\left(\frac{x}{\varepsilon}\right) = \partial_x (K \partial_x u) + \partial_y (K \partial_y u) \\ u(x,y,0) = u_0(x,y) \end{cases}$$

where $K(x,\varepsilon) = A + G\left(\frac{x}{\varepsilon}\right)$.

Let $z = \frac{x}{\varepsilon}$, then $\partial_x \mapsto \partial_x + \frac{1}{\varepsilon}\partial_z$

Ansantz: $u = \bar{u} + \varepsilon u_1 + \dots$

$$u_t = \left(\partial_x + \tfrac{1}{\varepsilon}\partial_z\right)\left(K\left(\partial_x + \tfrac{1}{\varepsilon}\partial_z\right)u\right) + \partial_y (K \partial_y u) - \tfrac{1}{\varepsilon} v(z)\,\partial_y u$$

$O\left(\frac{1}{\varepsilon^2}\right)$: $\partial_z (K \partial_z \bar{u}) = 0 \quad \Rightarrow \quad \bar{u}$ indep z

$O\left(\frac{1}{\varepsilon}\right)$: $\partial_z (K \partial_z u_1) + \partial_z (K \partial_x \bar{u}) + \underbrace{\cancel{\partial_x (K \partial_z \bar{u})}}_{\bar{u} \text{ indep } z} \to 0 \; - \; v(z)\,\partial_y \bar{u} = 0$

$$\langle \partial_z [K(\partial_z u_1 + \partial_x \bar{u})] \rangle_z - \cancel{\langle v(z)\, \partial_y \bar{u} \rangle_z} \to 0 = 0$$

periodic $\quad$ indep z

$$K(\partial_z u_1 + \partial_x \bar{u}) = A \leftarrow \text{indep } z$$

$$\underbrace{\cancel{\langle \partial_z u_1 \rangle_z}}_{FA} \to 0 + \langle \partial_x \bar{u} \rangle_z = \left\langle \frac{A}{K} \right\rangle_z$$

$$\partial_x \bar{u} = \frac{A}{\langle K \rangle_z^h} \quad \Rightarrow \quad A = \langle K \rangle_z^h \, \partial_x \bar{u}$$

$$K(\partial_z u_1 + \partial_x \bar{u}) = \langle K \rangle_z^h \, \partial_x \bar{u}$$

Also, u_1 indep of z $\because$ RHS is

$$O(1):\ \partial_z(K\partial_z u_2) + \partial_z(K\partial_x u_1) + \partial_x(K\partial_z u_1) - v(z)\partial_y u_1$$

$$+ \partial_x(K\partial_x \bar{u}) + \partial_y(K\partial_y \bar{u}) = \bar{u}_t$$

$$\underbrace{\cancel{\langle \partial_z(K(\partial_z u_2 + \partial_x u_1))\rangle_z}}_{\text{FA}} \to 0 \quad - \quad \cancel{\langle \underbrace{v(z)}_{\text{periodic}}\partial_y \underbrace{u_1}_{\text{indep } z}\rangle_z} \to 0$$

$$+\langle \partial_x(\underbrace{K(\partial_z u_1 + \partial_x \bar{u})}_{\langle K\rangle_z^h \partial_x \bar{u}})\rangle_z + \langle \partial_y(K\partial_y \bar{u})\rangle_z = \bar{u}_t$$

$$\bar{u}_t = \underbrace{\langle K\rangle_z^h}_{\alpha} \bar{u}_{xx} + \underbrace{\langle K\rangle_z}_{\beta} \bar{u}_{yy}$$

$$\boxed{\bar{u} = \mathcal{F}^{-1}\left\{ e^{(\alpha\chi^2 + \beta\eta)t} \right\}}$$

where $\mathcal{F}(\bar{u}_{xx}) = \chi^2$

and $\mathcal{F}(\bar{u}_{yy}) = \eta^2$

Problem 7

Aug 2017

a) $\varepsilon z^7 + (z-5)^2 = \varepsilon z$ (1)

Unpreturbed root:

$$(z-5)^2 = 0$$

$$z = 5$$

$$\Rightarrow z_0 = \sum_{n=0}^{\infty} a_n \varepsilon^{n/2} \quad \text{where} \quad a_0 = 5$$

Plug z_0 into (1):

$$\varepsilon(5 + a_1\varepsilon^{1/2})^7 + (a_1\varepsilon^{1/2})^2 = \varepsilon(5 + a_1\varepsilon^{1/2})$$

$O(\varepsilon)$: $5^7 + a_1^2 = 5$

$$a_1 = \pm\sqrt{5-5^7} = \pm i\sqrt{5^7-5}$$

$$\Rightarrow \boxed{\begin{aligned} z_1 &\sim 5 + i\sqrt{5^7-5}\,\varepsilon^{1/2} \\ z_2 &\sim 5 - i\sqrt{5^7-5}\,\varepsilon^{1/2} \end{aligned}}$$

Let $x = \varepsilon^{\alpha} z \Rightarrow z = \varepsilon^{-\alpha} x$ and plug into (1)

$$\varepsilon^{1-7\alpha} x^7 + (\varepsilon^{-\alpha} x - 5)^2 = \varepsilon^{1-\alpha} x$$

$$\varepsilon^{1-7\alpha} x^7 + \varepsilon^{-2\alpha} x^2 - 10\varepsilon^{-\alpha} x + 25 = \varepsilon^{1-\alpha} x$$

$$1 - 7\alpha = -2\alpha$$

$$\alpha = 1/5$$

$$x^7 + x^2 - 10\varepsilon^{1/5} x + 25\varepsilon^{2/5} - \varepsilon^{6/5} x = 0 \qquad (2)$$

Consider the unpreturbed version of (2):

$$x^7 + x^2 = 0$$

$$x^2(x^5+1)=0 \qquad\qquad \omega = -e^{i\frac{2\pi}{5}}$$

$$x = 0,\ 0,\ \omega,\ \omega^2,\ \omega^3,\ \omega^4,\ -1$$

correspond to z_1 and z_2

$$x_{k+2} = \omega^k + \sum_{n=1}^{\infty} b_n^k \varepsilon^{n/5} \qquad k = 1, \dots, 5$$

Plug x_{k+2} into (2)

$$(\omega + b_1^k \varepsilon^{1/5})^7 + (\omega^k + b_1^k \varepsilon^{1/5})^2 - 10\varepsilon^{1/5}(\omega^k + b_1^k \varepsilon^{1/5}) + 25\varepsilon^{2/5} - \varepsilon^{6/5}(\omega^k + b_1^k) = 0$$

$O(1)$: $\omega^{7k} + \omega^{2k} = 0$ ✓

$O(\varepsilon^{1/5})$: $\omega^{\cancel{6k}\,5k} b_1^k + 2\omega^k b_1^k - 10\omega^k = 0$

$$b_1^k = \frac{10}{\omega^{5k}+2}$$

$$b_1^k = \frac{10}{-\omega^k+2}$$

Recall: $z_j = x_j \varepsilon^{-1/5}$

$$\boxed{\begin{aligned} z_3 &\sim \frac{\omega}{\varepsilon^{1/5}} + \frac{10}{2-\omega} \\ z_4 &\sim \frac{\omega^2}{\varepsilon^{1/5}} + \frac{10}{2-\omega^2} \\ &\vdots \\ z_7 &\sim -\frac{1}{\varepsilon^{1/5}} + \frac{10}{3} \end{aligned}}$$

b) $x^2 = \varepsilon e^{x^4}$ (1)

$x^2 = \varepsilon(1 + x^4)$ (2)

Unpreturbed: $x^2 = 0$

$x = 0$

$$\Rightarrow x_0 = \sum_{n=1}^{\infty} a_n \varepsilon^{n/2} = a_1 \varepsilon^{1/2} + a_2 \varepsilon + a_3 \varepsilon^{3/2} + \dots$$

Plug x_0 into (2) and find a hierarchy of eqns:

$O(1)$: $0 = 0$

$O(\varepsilon^{1/2})$: $0 = 0$

$O(\varepsilon)$: $a_1^2 = 1$

$a_1 = \pm 1$

$O(\varepsilon^{3/2})$: $2a_1 a_2 = 0$

$a_2 = 0$

$O(\varepsilon^2)$: $2a_1 a_3 + a_2^2 = 0$

$a_3 = 0$

$O(\varepsilon^{5/2})$: $2a_1 a_4 + \cancel{2a_2 a_3} = 0$

$a_4 = 0$

$O(\varepsilon^3)$: $\cancel{2a_2 a_4} + 2a_1 a_5 + \cancel{a_3^2} = a_1^4$

$$a_5 = \frac{a_1^3}{2}$$

$$\boxed{\begin{aligned} x_1 &\sim \varepsilon^{1/2} + \tfrac{1}{2}\varepsilon^{5/2} \\ x_2 &\sim -\varepsilon^{1/2} - \tfrac{1}{2}\varepsilon^{5/2} \end{aligned}}$$

Problem 8 Aug 2017

a) Let $\Omega \subseteq \mathbb{C}$, $x_0 \in \Omega$ and f_n be a sequence of funcs.

Ptws: We say $f_n \to f$ conv ptws if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ st $\forall n > N \Rightarrow |f_n(x_0) - f(x_0)| < \varepsilon$.

Note: we are only concerned with fixed x_0 and N can vary.

Asymp: We say $f_N \to f$ conv asymp'ly for fixed N if $\forall \varepsilon > 0$ $\exists \delta > 0$ st $|x - x_0| < \delta \Rightarrow |f_N(x) - f(x)| < \varepsilon$.

Note: This time N is fixed and x can vary.

Ex: In each example I'll use ratio test to show conv.

i) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \left| \frac{x}{n+1} \right| \xrightarrow[x \text{ fixed}]{n \to \infty} 0 \quad \therefore \text{ ptws conv}$$

$$\xrightarrow[x \to \infty]{n \text{ fixed}} \infty \quad \therefore \text{ not asymp conv}$$

ii) $\sum_{n=0}^{\infty} \frac{n!}{x^n}$

$$\left| \frac{(n+1)!}{x^{n+1}} \cdot \frac{x^n}{n!} \right| = \left| \frac{n+1}{x} \right| \xrightarrow[x \text{ fixed}]{n \to \infty} \infty \quad \therefore \text{ not ptws conv}$$

$$\xrightarrow[x \to \infty]{n \text{ fixed}} 0 \quad \therefore \text{ asymp conv}$$

b) When f is uniformly continuous and integrable near x_0.