METHODS OF APPLIED MATHEMATICS COMPREHENSIVE EXAMINATION AUGUST 2018

Work on as many of the following problems as possible. Turn in *all* your work.

(1) Consider two point masses m_1 and m_2, respectively, joined by an ideal spring (Hooke's law, constant k, allowing the masses to pass through each other), in *collinear motion along the z-axis* of a cartesian frame in $\mathbb{R}^3$. Write the evolution equations given by Newton's second law, assuming each body is also subject to an overall central force field $F = -Gm_j/z^2$, with $z \neq 0$, and m_1, m_2 are the body's masses, and G is a constant.

(a) Identify the units of G

(b) Non-dimensionalize the equations of motion; identify non-dimensional parameters; Discuss all the possible dominant balances.

(c) Neglecting the central force field, find the solution of these equations corresponding to an initially stretched by an amount h state, with zero initial velocities.

(d) Write an asymptotic expansion for the leading order plus first correction of the solution assuming the initial stretching h is small compared to each mass's distance from the center of attraction at $z = 0$. Identify the small non-dimensional parameter.

(e) Sketch the leading order plus correction solutions and note their time scales of validity.

(2) Consider the eigenvalue problem on the interval $[-L, L]$

$$\epsilon y'' - (U(x) + \lambda)y = 0, \qquad y(-L) = 0, \frac{dy}{dx}(L) = 0,$$

with the potential:

$$U(x) = a\,\delta(x)$$

where $\delta(x)$ is the Dirac delta function.

(a) Discuss allowable values of the spectrum, λ, and allowable values of the parameter, a, to have a solution.

(b) Calculate the eigenvalue(s) and eigenfunctions as $\epsilon \to 0$. How does this result compare with the free space case? Third,

(c) Discuss the case with $L \sim \epsilon^b$, for $b > 0$. Is there a critical distinguished limit?

(3) Consider the boundary value problem

$$\epsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \epsilon u^2\,,$$

$$u(0, y) = g(y), \quad u(1, y) = 0, \quad u(x, 0) = f(x), \quad u(x, 1) = 0\,,$$

in $(x, y) \in [0, 1] \times [0, 1]$, for some functions $f(x)$, $g(y)$. The functions f and g can be taken to be smooth but are otherwise generic (i.e., do not take special values in $[0, 1]$).

(a) Find the outer solution when $\epsilon = 0$.

(b) Identify whether boundary layers exist, and if so, where they are located as $\epsilon \to 0$.

(c) Find a uniformly valid leading order asymptotic expression for the solution.

(4) Evaluate the following real integral using complex contour integration techniques, assuming $n = 2, 3, 4, \ldots$.

$$I = \int_0^\infty \frac{\sqrt{x}}{1+x^n}\,dx$$

(5) Find the first term in the asymptotic expansion of

Justify why it localizes

$$f(x) = \int_C \frac{1}{\zeta^5} \exp\left[i\,x \left(\zeta^8 - \frac{1}{\zeta^8} \right) \right] d\zeta$$

as $x \to +\infty$, where C is a closed contour around the origin.

(6) Consider the following integral:

$$\int_{-\infty}^{\infty} \frac{\exp(iky)\exp(-ik^3)}{\sqrt{\cosh(ak)}}\,dk$$

(a) For $a = 0$, find the leading order asymptotic expansion as $y \to +\infty$.
(b) Propose branch cuts for the case $a \neq 0$ so that the integral is well defined in the complex k plane.
(c) Find the leading order asymptotic expansion as $y \to +\infty$ for $a \neq 0$ (hint: utilize the periodicity of cosh in the complex k plane).

(7) (a) Find two term asymptotic expansions as $\epsilon \to 0$ for all roots of the equation

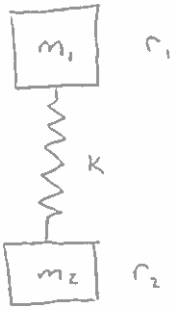
$$\epsilon x^3 + (1-\epsilon)x^2 - 2x + 1 = 0$$

(b) Find two term asymptotic expansions for the positive roots of

$$x^4 e^{-x^4} = \epsilon$$

(8) (a) Explain the difference between pointwise convergence and asymptotic convergence. Illustrate with the particular example of power series.
(b) As $x \to \infty$, if $f(x) \sim g(x)$, and $g(x) \sim h(x)$, is $f(x) \sim h(x)$? Provide a counter-example, or give a proof.

Problem 1



$$\begin{cases} m_1 \ddot{r}_1 = -\dfrac{Gm_1}{r_1^2} - k(r_1 - r_2) & (1) \\[2ex] m_2 \ddot{r}_2 = -\dfrac{Gm_2}{r_2^2} + k(r_1 - r_2) & (2) \end{cases}$$

- $r = 0$

a) $$\left.\begin{aligned} [m\ddot{r}] &= \frac{kg \cdot m}{s^2} \\ \left[\frac{G m_1 m_2}{r^2}\right] &= \frac{[G]\, kg^2}{m^2} \end{aligned}\right\} \Rightarrow [G] = \frac{kg\ m}{s^2} \cdot \frac{m^2}{kg} = \frac{m^3}{s^2}$$

b) Let $L = r_2(0)$ and $T = \sqrt{\frac{m_2}{k}}$ and define $r = zL$ and $t = \tau T$ where z and τ are non-dim. Depending on the size of the masses relative to each other, we might want to pick $T = \sqrt{\frac{m_1}{k}}$ to ensure a small parameter pops out of the non-dim'zation.

$$\dot{r} = \frac{dr}{dt} = \frac{d(Lz)}{dt} = L \frac{dz}{d\tau}\frac{d\tau}{dt} = \frac{L}{T}\dot{z}$$

$$\ddot{r} = \frac{L}{T^2}\ddot{z}$$

I will non-dim (1) first.

$$(1) \Rightarrow \frac{m_1 L}{T^2} \ddot{z}_1 = -\frac{G m_1}{L^2 (z_1)^2} - kL(z_1 - z_2)$$

$$\ddot{z}_1 = -\frac{G T^2}{L^3 (z_1)^2} - \frac{k T^2}{m_1}(z_1 - z_2)$$

Non Dim Parameters

$$\gamma = \frac{G T^2}{L^3}$$

$$\alpha = \frac{k T^2}{m_1} = \frac{m_2}{m_1}$$

$$z_1(0) = \frac{r_1(0)}{L} = \frac{r_2(0)+h}{L} = 1 + \varepsilon$$

$$z_2(0) = \frac{r_2(0)}{L} = 1$$

$$\dot{z}(0) = \frac{L \dot{r}(0)}{T} = 0$$

Similarly for (2) so

$$\Rightarrow \begin{cases} \ddot{z}_1 = -\dfrac{\gamma}{z_1^2} - \alpha(z_1 - z_2) & (3) \\ \ddot{z}_2 = -\dfrac{\gamma}{z_2^2} + (z_1 - z_2) & (4) \end{cases}$$

If $m_2 \ll m_1$ then α is a small parameter. If $m_1 \ll m_2$ we would have chosen T differently and our small parameter $\alpha = \frac{m_1}{m_2}$ would have occured in (4). Other than m_1 and m_2, we could have $z_2(0) \ll z_1(0)$ or $z_1(0) - z_2(0) \ll z_1(0)$.

c) Neglecting F we have the following system:

$$\begin{cases} \ddot{z}_1 = -\alpha(z_1 - z_2) & (5) \\ \ddot{z}_2 = z_1 - z_2 & (6) \\ z_1(0) - z_2(0) = \varepsilon \\ \dot{z}_1(0) = \dot{z}_2(0) = 0 \end{cases}$$

Method 1: Solve for displacement relative to each other. #1

If we neglect F, the problem essentially turns into two masses oscillating relative to each other so define $x = z_1 - z_2$.

$$\ddot{x} = \ddot{z}_1 - \ddot{z}_2 = -\alpha(z_1 - z_2) - (z_1 - z_2) = -\alpha x - x = -(1+\alpha)x$$

$$\ddot{x} = (1+\alpha)x$$

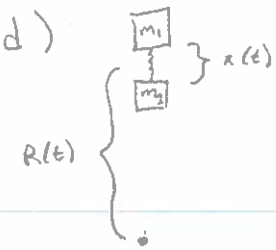
$$x = A\cos(\sqrt{1+\alpha}\,\tau) + B\sin(\sqrt{1+\alpha}\,\tau)$$

But $\dot{x}(0) = \dot{z}_1(0) - \dot{z}_2(0) = 0$ so

$$\begin{cases} x(0) = A = z_1(0) - z_2(0) = h \\ \dot{x}(0) = B\sqrt{1+\alpha} = 0 \end{cases}$$

$$\boxed{x(\tau) = h\cos(\sqrt{1+\alpha}\,\tau)}$$

d)



$$\begin{cases} Z_1 = R + \frac{x}{2} \\ Z_2 = R - \frac{x}{2} \end{cases} \Rightarrow \begin{cases} R = \frac{Z_1 + Z_2}{2} \\ x = Z_1 - Z_2 \\ x(0) = h \\ R(0) = R_0 \end{cases}$$

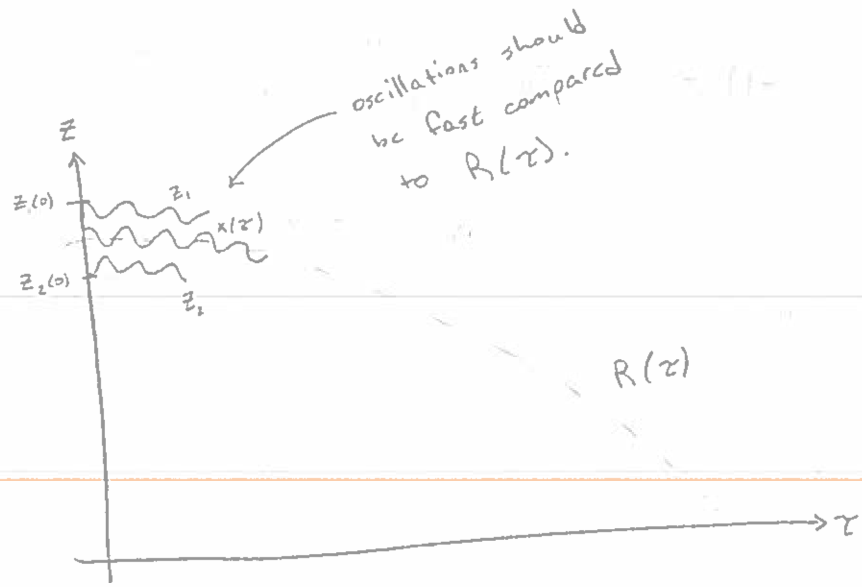
$$\ddot{R} = -\frac{1}{2}\left(\frac{C_1}{Z_1^2} + \frac{C_2}{Z_2^2}\right) + \left(\frac{1-\alpha}{2}\right)(Z_1 - Z_2)$$

$$\ddot{R} = -\frac{1}{2}\left(\frac{C_1}{(R+\frac{x}{2})^2} + \frac{C_2}{(R-\frac{x}{2})^2}\right) + \left(\frac{1-\alpha}{2}\right)x$$

Since $R_0 \gg h$, the masses can basically be considered as one b/c the oscillations won't affect the general path of the gravitational pull. And as we found earlier, $x(\tau) \leq h$

Need to expand around h.

e)



Method 2 : Laplace Transform and solve system of eqn

$$\mathcal{L}(z) = \hat{z}$$

$$\mathcal{L}(\ddot{z}) = \int_0^\infty e^{-s\tau}\ddot{z}\,d\tau = \cancelto{0}{e^{-s\tau}\dot{z}\Big|_0^\infty} + s\int_0^\infty e^{-s\tau}\dot{z}\,d\tau = \cancelto{-sz(0)}{se^{-s\tau}z\Big|_0^\infty} + s^2\int_0^\infty e^{-s\tau}z\,d\tau$$

$$= s^2\hat{z} - sz(0)$$

$$\begin{cases} s^2\hat{z}_1 = -\alpha(\hat{z}_1 - \hat{z}_2) + sz_1(0) & (7) \\ s^2\hat{z}_2 = \hat{z}_1 - \hat{z}_2 + sz_2(0) & (8) \end{cases}$$

$$\Rightarrow \hat{z}_2 = \frac{1}{s^2+1}\left(\hat{z}_1 + sz_2(0)\right) \qquad (9)$$

Sub (9) into (7)

$$\rightarrow \quad s^2\hat{z}_1 = -\alpha\left(\hat{z}_1 - \frac{\hat{z}_1 + sz_2(0)}{s^2+1}\right) + sz_1(0)$$

$$s^{\cancel{2}}(s^2+1)\hat{z}_1 = -\alpha\left(s^{\cancel{2}}\hat{z}_1 + \cancel{s}z_2(0)\right) + \cancel{s}z_1(0)$$

$$\hat{z}_1 = \frac{(s^2+1)z_1(0) - \alpha z_2(0)}{s^3 + (1+\alpha)s} = \frac{A}{s} + \frac{Bs}{s^2+(1+\alpha)}$$

$$\text{where} \quad A = \frac{z_1(0) - \alpha z_2(0)}{1+\alpha} \quad \text{and} \quad B = \frac{2z_1(0) + \alpha\varepsilon}{1+\alpha}$$

$$z_1 = \int_{c-i\infty}^{c+i\infty}\left(\frac{A}{s} + \frac{Bs}{s^2+(1+\alpha)}\right)e^{s\tau}\,ds = A + B\cos\left(\sqrt{1+\alpha}\,\tau\right)$$

$$z_1(\tau) = \frac{z_1(0) - \alpha z_2(0)}{1+\alpha} + \frac{2z_1(0) + \alpha\varepsilon}{1+\alpha}\cos\left(\sqrt{1+\alpha}\,\tau\right)$$

$$\boxed{z_1(\tau) = \frac{1+\varepsilon-\alpha}{1+\alpha} + \frac{2+(2+\alpha)\varepsilon}{1+\alpha}\cos\left(\sqrt{1+\alpha}\,\tau\right)}$$

Similarly we solve for z_2 by plugging $\hat{z}_1 = \dfrac{\alpha \hat{z}_2 - s z_1(0)}{s^2+\alpha}$ into (8):

$$s^2 \hat{z}_2 = \frac{\alpha \hat{z}_2 - s z_1(0)}{s^2+\alpha} - \hat{z}_2 - s z_2(0)$$

$$s^{\cancel{2}}(s^2+\alpha)\hat{z}_2 = -\cancel{s} z_1(0) - s^{\cancel{2}}\hat{z}_2 - \cancel{s}(s^2+\alpha) z_2(0)$$

$$\hat{z}_2 = -\frac{(s^2+\alpha) z_2(0) + z_1(0)}{s(s^2+(\alpha+1))} = \frac{A}{s} + \frac{Bs+C}{s^2+(\alpha+1)}$$

$$-s^2 z_2(0) - (\alpha z_2(0) + z_1(0)) = A s^2 + A(\alpha+1) + B s^2 + C s$$

$$\begin{cases} A(\alpha+1) = -(\alpha z_2(0) + z_1(0)) \\ C = 0 \\ A + B = -z_2(0) \end{cases}$$

$$B = \frac{-\cancel{\alpha z_2(0)} - z_2(0) + \cancel{\alpha z_2(0)} + z_1(0)}{\alpha+1}$$

$$A = -\frac{\alpha z_2(0) + z_1(0)}{1+\alpha} = -\frac{\alpha+1+\varepsilon}{1+\alpha} \qquad B = \frac{\varepsilon}{1+\alpha}$$

$$z_2 = \int_{c-i\infty}^{c+i\infty} e^{s\tau}\left(\frac{A}{s} + \frac{Bs}{s^2+(1+\alpha)}\right) ds = A + B\cos(\sqrt{1+\alpha}\,\tau)$$

$$\boxed{z_2 = -\left(1 + \frac{\varepsilon}{1+\alpha}\right) + \frac{\varepsilon}{\alpha+1}\cos(\sqrt{1+\alpha}\,\tau)}$$

Preturb using γ as small parameter

d) $$\begin{cases} \ddot{z}_1 = -\frac{\gamma}{z_1^2} - \alpha(z_1 - z_2) \\ \ddot{z}_2 = -\frac{\gamma}{z_2^2} + (z_1 - z_2) \end{cases}$$

Recall $\gamma = \frac{GT^2}{L^3}$. Since $h \ll r_2(0)$ and $h \ll r_1(0)$ and $L = r_2(0)$, so we can call γ our small parameter.

Ansantz:

$$z_i \sim z_i^{(0)} + z_i^{(1)}\gamma + \dots \quad \text{as } \tau \to \infty$$

$\mathcal{O}(1)$: $$\begin{cases} \ddot{z}_1^{(0)} = -\alpha(z_1^{(0)} - z_2^{(0)}) \\ \ddot{z}_2^{(0)} = (z_1^{(0)} - z_2^{(0)}) \end{cases}$$

This system was solved in part c

$\mathcal{O}(\gamma)$: $$\begin{cases} \ddot{z}_1^{(1)} = -\frac{1}{(z_1^{(0)})^2} - \alpha(z_1^{(1)} - z_2^{(1)}) \\ \ddot{z}_2^{(1)} = -\frac{1}{(z_2^{(0)})} + (z_1^{(1)} - z_2^{(1)}) \end{cases}$$

$$\frac{1}{(\ddot{z}_i^{(0)})^2} = \frac{1}{(A + B(\tau))^2} \sim A^2 - 2AB(\tau)$$

$$\sim \begin{cases} \ddot{z}_1^{(1)} = -\alpha(z_1^{(1)} - z_2^{(1)}) - \underbrace{\left(\frac{1-\alpha}{1+\alpha}\right)^2}_{=a} \\ \ddot{z}_2^{(1)} = z_1^{(1)} - z_2^{(1)} - (-1)^2 \end{cases}$$

$$= \begin{cases} \ddot{z}_1^{(1)} = -\alpha(z_1^{(1)} - z_2^{(1)}) - a \\ \ddot{z}_2^{(1)} = (z_1^{(1)} - z_2^{(1)}) - 1 \end{cases}$$

Take the Laplace Trans:

$$\begin{cases} s^2 \hat{z}_1 = -\alpha(\hat{z}_1 - \hat{z}_2) + \quad (9) \\ s^2 \hat{z}_2 = (\hat{z}_1 - \hat{z}_2) + \cancel{s} - \frac{1}{s} \quad (10) \end{cases}$$

$$\Rightarrow \hat{z}_2 = \frac{1}{s^2+1}\left(\hat{z}_1 + \cancel{s} - \frac{1}{s}\right) \qquad (11)$$

Plug (11) into (9):

$$s^2 \hat{z}_1 = -\alpha\left(\hat{z}_1 - \frac{1}{s^2+1}\left(\hat{z}_1 + \cancel{s} - \frac{1}{s}\right)\right) + s(1+\cancel{\varepsilon}) - \frac{a}{s}$$

$$\left(s^3(s^2+1) + \alpha s(s^2+1) - \alpha s\right)\hat{z}_1 = \cancel{\quad} - \alpha + \cancel{s^2(s^2+1)(1+\varepsilon)} - a(s^2+1)$$

$$\left(s^2(s^2+1) + \alpha s^3\right)\hat{z}_1 = \cancel{\alpha s^2} - \alpha + \cancel{(s^4+s^2)(1+\varepsilon)} - a(s^2+1)$$

$$\hat{z}_1 = \frac{(1+\varepsilon)s^4 + (\alpha + \varepsilon + 1 - a)s^2 - \alpha - a}{s^3(s^2+1+\alpha)} \qquad \alpha + \frac{1-\alpha}{1+\alpha} = \frac{\alpha^2+1}{1+\alpha}$$

$$z_1^{(1)} = -\frac{t^2(\alpha+a)}{2(1+\alpha)} + \frac{\left(e^{-t\sqrt{-1-\alpha}} + e^{t\sqrt{-1-\alpha}}\right)\alpha(a+\alpha)}{2(1+\alpha)^2} + \frac{1+3\alpha - a\alpha + \alpha^2 + \varepsilon + \alpha\varepsilon}{(1+\alpha)^2}$$

$$* \quad z_1^{(1)} = \frac{1}{(1+\alpha)^2}\left(-\frac{(\alpha^2+1)}{2}t^2 + \frac{1+\alpha^2}{1+\alpha}\underbrace{\cos\left(\sqrt{-1-\alpha}\,t\right)}_{\cosh(\sqrt{1+\alpha}\,t)} + 1 + 3\alpha - \frac{\alpha-\alpha^2}{1+\alpha} + \alpha^2 + \varepsilon(1+\alpha)\right),$$

Simillerly, we solve for $z_2^{(1)}$ by plugging $\hat{z}_1 = \frac{1}{s^2+\alpha}\left(\alpha\hat{z}_2 + s(1+\varepsilon) - \frac{a}{s}\right)$ into (10):

$$s^2 \hat{z}_2 = \frac{1}{s^2+\alpha}\left(\alpha\hat{z}_2 + s(1+\varepsilon) - \frac{a}{s}\right) - \hat{z}_2 + s - \frac{1}{s}$$

Take the Laplace Trans:

$$\begin{cases} s^2 \hat{z}_1 = -\alpha(\hat{z}_1 - \hat{z}_2) - \frac{a}{s} & (9) \\ s^2 \hat{z}_2 = \hat{z}_1 - \hat{z}_2 - 1/s & (10) \end{cases}$$

$$\Rightarrow \hat{z}_2 = \frac{1}{s^2+1}(\hat{z}_1 - 1/s) \qquad (11)$$

Plug (11) into (9):

$$s^2 \hat{z}_1 = -\alpha\left(\hat{z}_1 - \frac{1}{s^2+1}\left(\hat{z}_1 - \frac{1}{s}\right)\right) - \frac{a}{s}$$

$$\left(s^3(s^2+1) + \alpha s(s^2+1) - \alpha s\right)\hat{z}_1 = -\alpha - a(s^2+1)$$

$$\hat{z}_1 = -\frac{\alpha + a(s^2+1)}{s^3(s^2+1+\alpha)}$$

$$z_1^{(1)} = \frac{a+\alpha}{(1+\alpha)^2}\left(-\frac{\tau^2}{2} - \frac{1}{1+\alpha}\cos(\sqrt{1+\alpha}\,\tau) + \frac{1}{1+\alpha}\right)$$

Simillerly we solve for $z_2^{(1)}$ by plugging $\hat{z}_1 = \frac{1}{s^2+\alpha}\left(\alpha z_2 - \frac{a}{s}\right)$ into (10):

$$s^2 \hat{z}_2 = \frac{1}{s^2+\alpha}\left(\alpha \hat{z}_2 - \frac{a}{s}\right) - \hat{z}_2 - 1/s$$

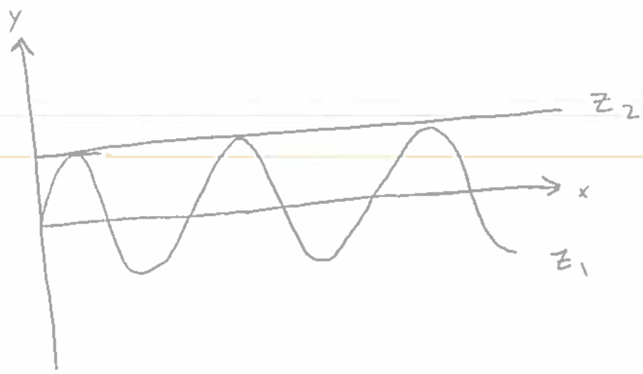
$$\left(s^3(s^2+\alpha) - \cancel{s\alpha} + s(s^2+\cancel{\alpha})\right)\hat{z}_2 = -a - (s^2+\alpha)$$

$$\hat{z}_2 = -\frac{a + s^2 + \alpha}{s^3(s^2+1+\alpha)}$$

$$z_2^{(1)} = -\frac{\tau^2(a+\alpha)}{2(1+\alpha)} - \frac{a-1}{(1+\alpha)^2}\cos(\sqrt{1+\alpha}\,\tau) + \frac{a-1}{(1+\alpha)^2}$$

$$\begin{cases} Z_1 \sim \overset{\text{part c}}{Z_1^{(0)}} + \gamma \overset{*}{Z_1^{(1)}} \\[2ex] Z_2 \sim \underset{\text{part c}}{Z_2^{(0)}} + \gamma \underset{**}{Z_2^{(1)}} \end{cases} \quad \text{as } \tau \to \infty$$

e)



$\gamma = 10^{-7}$

$\varepsilon = 0.001$

$\alpha = 1 \quad (m_1 = m_2)$

Preturb using ε as small parameter

d) $$\begin{cases} \ddot{z}_1 = -\dfrac{\gamma}{z_1^2} - \alpha(z_1 - z_2) \\ \ddot{z}_2 = -\dfrac{\gamma}{z_2^2} + z_1 - z_2 \end{cases}$$

Recall $\varepsilon = \dfrac{h}{r_2(0)} \ll 1$.

Anzants: $z_i \sim z_i^{(0)} + z_i^{(1)}\varepsilon + \ldots$

$$\ddot{z}_1^{(0)} + \ddot{z}_1^{(1)}\varepsilon = -\frac{\gamma}{(z_1^{(0)} + z_1^{(1)}\varepsilon + \ldots)^2} - \alpha\left(z_1^{(0)} + \varepsilon z_1^{(1)} + \ldots - (z_2^{(0)} + z_2^{(1)}\varepsilon + \ldots)\right)$$

$$\begin{cases} \ddot{z}_1^{(0)} + \ddot{z}_1^{(1)}\varepsilon + \ldots = -\gamma + 2\gamma(-1 + z_1^{(0)} + z_1^{(1)}\varepsilon + \ldots) - \alpha(z_1^{(0)} + \varepsilon z_1^{(1)} + \ldots - (z_2^{(0)} + z_2^{(1)}\varepsilon + \ldots)) \\ \ddot{z}_2^{(0)} + \ddot{z}_2^{(1)}\varepsilon + \ldots \approx -\gamma + 2\gamma(-1 + z_2^{(0)} + z_1^{(1)}\varepsilon + \ldots) + z_1^{(0)} + \varepsilon z_1^{(1)} + \ldots - (z_2^{(0)} + z_2^{(1)}\varepsilon + \ldots) \end{cases}$$

$\mathcal{O}(1)$: $$\begin{cases} \ddot{z}_1^{(0)} = -\gamma + 2\gamma(z_1^{(0)} - 1) - \alpha(z_1^{(0)} - z_2^{(0)}) \\ \ddot{z}_2^{(0)} = -\gamma + 2\gamma(z_1^{(0)} - 1) + z_1^{(0)} - z_2^{(0)} \end{cases}$$

If we say γ is small enough to preturb $z_i^{(0)}$ with γ and get the same system as in part (a). We can do this because $\gamma = \dfrac{GT^2}{L^3}$ and L is relatively large.

Thus $z_1^{(0)}$ and $z_2^{(0)}$ are the same as z_1 and z_2 from part (a).

$\mathcal{O}(\varepsilon)$: This system will take some more effort to solve. But it can be done by setting up a matrix to represent the system and using the eigenvalues or maybe by using the Laplace transform again.

Problem 2

Fall 2018

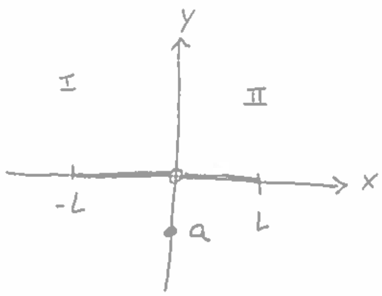
$$\varepsilon y'' - (u(x) + \lambda) y = 0 \qquad (1)$$

$$y'' = \frac{u(x)+\lambda}{\varepsilon} y$$

$y(-L) = 0$

$y'(L) = 0$

$u(x) = a\delta(x)$



a) We need $a < 0$ for physical relevance and no restrictions need to placed on λ because we have a finite domain. If we want to send L to infinity, we'll need $\frac{u(x)+\lambda}{\varepsilon} > 0$ in Regions I and II to ensure our soln decays.

For the rest of this problem, we'll assume $\frac{u(x)+\lambda}{\varepsilon} > 0$ in I + II so that we can find a soln. $u(x) = 0$ in I + II so $\lambda > 0$.

b) Regions I + II $\quad y_i'' = \underbrace{\frac{\lambda}{\varepsilon}}_{>0} y_i = \beta^2 y_i$

$$y_i = A_i e^{-x\beta} + B_i e^{x\beta}$$

$$y_1(-L) = A_1 e^{L\beta} + B_1 e^{-L\beta} = 0$$

$$\underline{B_1 = -A_1 e^{2L\beta}}$$

$$y_2'(L) = -A_2 \beta e^{-L\beta} + B_2 \beta e^{L\beta} = 0$$

$$B_2 = A_2 e^{-2L\beta}$$

Patching: $\quad y_1(0) = y_2(0)$

$$A_1 + B_1 = -A_2 \beta + B_2 \beta$$

$$A_1 - A_1 e^{2L\beta} = A_2 \beta \left(e^{-2L\beta} - 1 \right)$$

$$A_1 = A_2 \beta \left(\frac{e^{-2L\beta} - 1}{1 - e^{2L\beta}} \right) = B_2 \beta$$

$$A_2 = \frac{A_1}{\beta} \left(\frac{1 - e^{2L\beta}}{e^{-2L\beta} - 1} \right) \quad \text{or} \quad \underline{B_2 = \frac{A_1}{\beta}}$$

$$\underline{A_2 = \frac{A_1}{\beta} e^{2L\beta}}$$

Let $0 < \delta \ll 1$

$$\int_{-\delta}^{\delta} y'' = \int_{-\delta}^{\delta} \frac{u(x)+\lambda}{\varepsilon}\, y$$

$$y_2'(\delta) - y_1'(-\delta) = \int_{-\delta}^{\delta} \frac{u(x)}{\varepsilon}\, y + \cancel{\int_{-\delta}^{\delta} \frac{\lambda}{\varepsilon}\, y} \to 0$$

As $\delta \to 0$:

$$y_2'(0) - y_1'(0) = \frac{a}{\varepsilon}\, y(0) = \frac{a}{\varepsilon}\, y_1(0)$$

$$-A_2\beta + B_2\beta + A_1\beta - B_1\beta = \frac{a}{\varepsilon}\left(A_1 - A_1 e^{2L\beta}\right)$$

$$-A_1\left(\frac{1-e^{2L\beta}}{e^{-2L\beta}-1}\right) + A_1 + A_1\beta + A_1\beta e^{2L\beta} = \frac{a}{\varepsilon} A_1\left(1 - e^{2L\beta}\right)$$

$$-\left(\frac{1-e^{2L\beta}}{e^{-2L\beta}-1}\right) + 1 + \beta\left(1 + e^{2L\beta}\right) = \frac{a}{\varepsilon}\left(1 - e^{2L\beta}\right)$$

$$-e^{2L\beta} + 1 + \beta\left(1 + e^{2L\beta}\right) = \frac{a}{\varepsilon}\left(1 - e^{2L\beta}\right)$$

$$\beta\left(\frac{1+e^{2L\beta}}{1-e^{2L\beta}}\right) = \frac{a}{\varepsilon} - 1$$

$$\sqrt{\frac{\lambda}{\varepsilon}}\left(\frac{1+e^{2L\sqrt{\lambda/\varepsilon}}}{1-e^{2L\sqrt{\lambda/\varepsilon}}}\right) = \frac{a}{\varepsilon} - 1$$

Solve for λ. Could maybe be done asymptotically after multiplying by ε or numerically with Newton's method or some such method.

b) sps $a \neq 0$

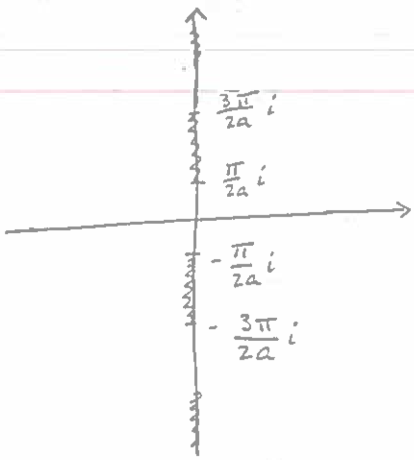
$\Rightarrow \cosh(ak) = 0$ is a b.p.

$\cos(iak) = 0$

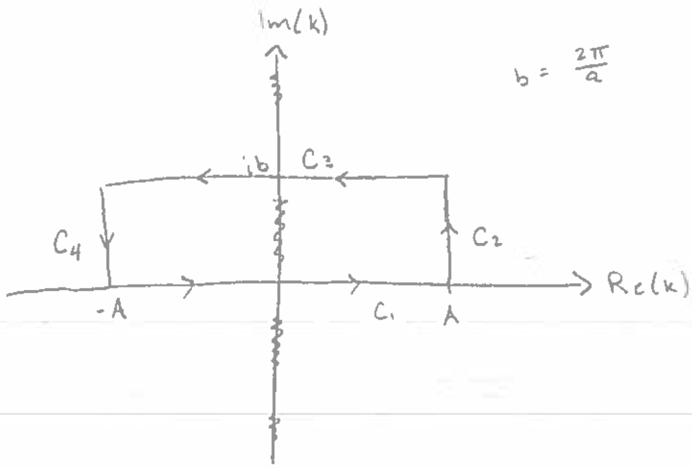
$iak = \frac{\pi}{2} + n\pi$

$k = \frac{1}{ia}\left(\frac{\pi}{2} + n\pi\right) = \ldots -\frac{3\pi}{2a}i, -\frac{\pi}{2a}i, \frac{\pi}{2a}i, \frac{3\pi}{2a}i, \ldots$

Branch Cut



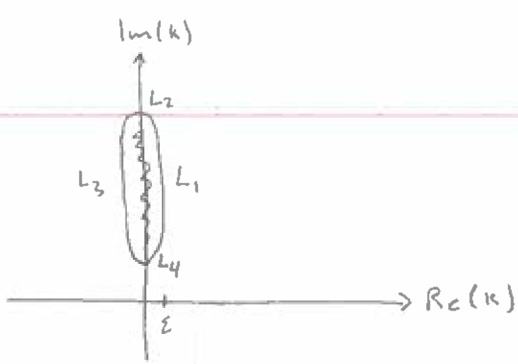
c)



$b = \frac{2\pi}{a}$

$C = \cup C_j$

Also



$\tilde{C} = \cup L_j$

$$\int_{\tilde{C}} = \int_C$$

C_2: $\displaystyle \left|I_{C_2}\right| = \left|\int_A^{A+ib} \frac{e^{iky} e^{-ik^3}}{\sqrt{\cosh(ak)}}\, dk\right| = \left| i e^{iAy} \int_0^b \frac{e^{-i(A+it)^3}}{\sqrt{\cosh(a(A+it))}} \underbrace{e^{-ty}}_{\substack{<1 \because y>0 \\ \frac{2\pi}{a} \geq t \geq 0}} dt \right|$

Let $K = A + it$

$dk = i\,dt$

$$\leq \int_0^b \frac{\left|e^{-i(A^3 + 3A^2ti - 3At^2 - it^3)}\right|}{\left|\sqrt{\cosh(aA + ait)}\right|}\, dt$$

$$\leq \int_0^b \frac{\left|e^{-3A^2t}\right| \left|e^{t^3}\right|}{\left|\frac{e^{aA+ait} + e^{-aA-ait}}{2}\right|^{1/2}}\, dt$$

$$\leq \sqrt{2} \int_0^b \frac{\left|e^{-3A^2t}\right| \left|e^{b^3}\right|}{\left|e^{aA+ait} + e^{-aA-ait}\right|}\, dt \xrightarrow[A\to\infty]{} \infty + 0$$

$A \to \infty$: $\longrightarrow 0$

$\longrightarrow 0$ as $A \to \infty$

C_4: Simillar to above

$\left|I_{C_4}\right| \longrightarrow 0$ as $A \to \infty$

C_3: $\displaystyle \int_{ib+\infty}^{ib-\infty} \frac{e^{iyk} e^{-ik^3}}{\sqrt{\cosh(ak)}}\, dk = \int_{\infty}^{-\infty} \frac{e^{iy(z+ib)} e^{-i(z+ib)^3}}{\sqrt{\cosh(az + 2\pi i)}}\, dz$

$K = z + ib$

$dk = dz$

$$= e^{-yb} \int_{\infty}^{-\infty} \frac{e^{iyz} e^{\overbrace{-i(z^3 + z^2bi - zb^2 - ib^3)}^{\sim z^3 \text{ as } z \to \pm\infty}}}{\sqrt{\cosh(az)}}\, dz$$

$$\sim e^{-yb} \int_{\infty}^{-\infty} \frac{e^{iyz} e^{-iz^3}}{\sqrt{\cosh(az)}}\, dz$$

$$\underline{L_1:}\quad \int_{\frac{i\pi}{2a}}^{\frac{i3\pi}{2a}} \frac{e^{iky - ik^3}}{\sqrt{\cosh(ak)}}\,dk = -i\int_{\pi/2a}^{3\pi/2a} \frac{e^{ty + t^3}}{\sqrt{\cos(at)}}\,dt$$

$t = ik$

$dt = i\,dk$

$$= -i\,y^{1/2}\int_{\pi/2a}^{3\pi/a} \frac{e^{y^{3/2}(x + x^3)}}{\sqrt{\cos(a\sqrt{y}\,x)}}\,dx$$

Let $t = x\sqrt{y}$

$h(x) = x + x^3$

$h'(x) = 1 + 3x^2$ $\qquad x_1 = \frac{i}{\sqrt{3}}\,,\quad x_2 = -\frac{i}{\sqrt{3}}$

$h''(x) = 6x$ $\qquad \left.\begin{aligned} h''(x_1) &= 2i\sqrt{3} \\ h''(x_2) &= -2i\sqrt{3}\end{aligned}\right\}$ contribute equally

$$\int_{L_1} \sim -i\,y^{1/2}\left(\cosh\left(a\sqrt{y/3}\right)\right)^{-1/2}\left(e^{y^{3/2}\overset{=\frac{2}{\sqrt{3}}i}{h(x_1)}} + e^{y^{3/2}\overset{=-\frac{2}{\sqrt{3}}i}{h(x_2)}}\right)\sqrt{\frac{2\pi}{y^{3/2}(2\sqrt{3})}} = \bar{L}$$

$$\underline{L_3:}\quad \int_{\frac{3i\pi}{2a}}^{\frac{i\pi}{2a}} \frac{e^{iyk}\,e^{-ik^3}}{\sqrt{\cosh(ak)}\,e^{2\pi i}}\,dk = -(-i)\int_{\frac{\pi}{2a}}^{3\pi/2a} \frac{e^{y^{3/2}(x + x^3)}}{e^{\pi i}\sqrt{\cos(a\sqrt{y}\,x)}}\,dk$$

$x\sqrt{y} = -ik$ $\qquad\qquad = \bar{L}$

$\sqrt{y}\,dx = -i\,dk$

L_2: $\left| \int_0^{\pi} \frac{e^{\varepsilon i e^{i\theta} - \varepsilon^3 i e^{i3\theta}}}{\sqrt{\cosh(a\varepsilon e^{i\theta})}} \varepsilon i e^{i\theta} \, d\theta \right| \le \varepsilon \int_0^{\pi} \frac{\overbrace{\left| e^{-\varepsilon \sin\theta + \varepsilon^3 \sin 3\theta} \right|}^{\xrightarrow{\varepsilon \to 0} 1}}{\underbrace{\left| \sqrt{\cosh(a\varepsilon e^{i\theta})} \right|}_{\xrightarrow{\varepsilon \to 0} 1}} d\theta$

$k = \varepsilon e^{i\theta}$

$dk = \varepsilon i e^{i\theta} d\theta$ $\longrightarrow 0$ as $\varepsilon \to 0$

L_4: Similler to above

$\int_{\pi}^{2\pi} \sim\sim \longrightarrow 0$ as $\varepsilon \to 0$

Recall $\int_C = \int_{\tilde{C}}$ so as $A \to \infty$ and $\varepsilon \to 0$ we have:

$$(1 - e^{-yb}) \int_{-\infty}^{\infty} = \int_{L_1} + \int_{L_3} \sim \overline{I} + \overline{I} \quad \text{as } y \to \infty$$

$$\int_{-\infty}^{\infty} \sim \frac{2\overline{I}}{1 - e^{-yb}} \quad \text{as } y \to \infty$$

Indeed, as $y \to \infty$, $e^{-yb} \to 0$ so

$$\boxed{\int_{-\infty}^{\infty} \frac{e^{iyk - ik^3}}{\sqrt{\cosh(ak)}} dk \sim -i y^{-1/4} \left(\cosh\left(a\sqrt{\tfrac{y}{3}}\right)\right)^{-1/2} \left(2\cos\left(y^{3/2} \tfrac{2}{\sqrt{3}}\right)\right) \sqrt{\tfrac{\pi}{\sqrt{3}}} \quad \text{as } y \to \infty}$$

Problem 6

$$\int_{-\infty}^{\infty} \frac{e^{iky-ik^3}}{\sqrt{\cosh(ak)}}\,dk$$

a) $a = 0$

$\cosh(ak) = \cos(iak) = \cos(0) = 1$

Let's assume we're working with the positive branch of $\sqrt{\ }$ so that we have $\sqrt{1} = 1$.

$$\Rightarrow I_1 = \int_{-\infty}^{\infty} e^{iky - ik^3}\,dk$$

Change of var: $k = y^\alpha z \Rightarrow dk = y^\alpha dz$

$$I_1 = \int_{-\infty}^{\infty} y^\alpha e^{iy^{\alpha+1}z - iy^{3\alpha}z^3}\,dz$$

$1+\alpha = 3\alpha$

$\alpha = 1/2$

$$= \int_{-\infty}^{\infty} y^{1/2} e^{iy^{3/2}(z - z^3)}\,dz$$

Let $h(z) = zi - z^3 i$

The integral will conv so long as $Re(h(z)) < 0$.

As $y \to \infty$, $z \to \infty$ also thus z^3 will dominate z.

So we need $Re(iz^3) > 0$

$$\Rightarrow Re(r^3 i(\cos 3\theta + i\sin 3\theta)) > 0$$

$$-\sin 3\theta > 0$$

$$\sin 3\theta < 0$$

$\sin(3\theta) < 0$

$\pi + 2k\pi < 3\theta < 2\pi + 2k\pi$

$\frac{\pi}{3} + \frac{2k}{3}\pi < \theta < \frac{2\pi}{3} + \frac{2k}{3}\pi$

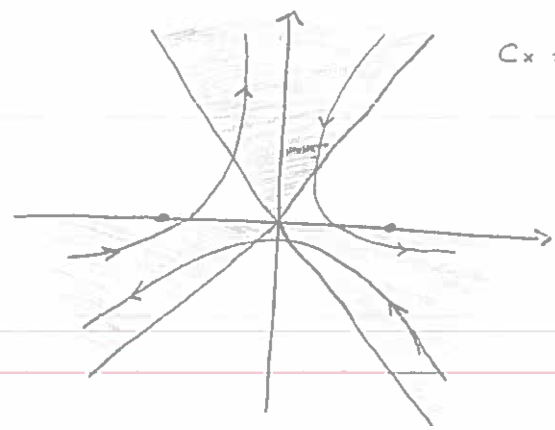
$\frac{\pi}{3} < \theta < \frac{2\pi}{3}$

$\pi < \theta < \frac{4\pi}{3}$

$\frac{5\pi}{3} < \theta < 2\pi$

(shaded) = region of conv

• = saddle point



Cx z

$h(z) = zi - z^3 i$

$h'(z) = i - 3z^2 i = 0$

$3z^2 = 1 \Rightarrow z^* = \pm \frac{1}{\sqrt{3}}$

$h(1/\sqrt{3}) = \left(\frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}}\right) i = \frac{2}{3\sqrt{3}} i$

$h(-1/\sqrt{3}) = -\frac{2}{3\sqrt{3}} i$

> no real parts so they contribute equally

$h''(z) = -6zi$

$n=2$

$\Rightarrow m = 0, 1$

$h''(1/\sqrt{3}) = -2i\sqrt{3} = 2\sqrt{3}\, e^{i\,3\pi/2}$ $\qquad \alpha_1 = \frac{3\pi}{2}$

$h''(-1/\sqrt{3}) = 2i\sqrt{3} = 2\sqrt{3}\, e^{i\pi/2}$ $\qquad \alpha_2 = \pi/2$

Need $Im(h(z)) = 0$ and $Re(h(z)) < 0$

$\underline{\alpha_1:}$ $\Rightarrow \sin(\alpha_1 + n\theta_m) = 0$

$\frac{3\pi}{2} + 2\theta_m = m\pi$

$2\theta_m = -\frac{3\pi}{2} + m\pi$

$\theta_m = -\frac{3\pi}{4} + \frac{m\pi}{2}$

$\theta_0 = -\frac{3\pi}{4}$

$\theta_1 = -\frac{\pi}{4}$

$\cos(\alpha_1 + n\theta_m) < 0$

$\theta_0: \frac{3\pi}{2} + 2\left(-\frac{3\pi}{4}\right) = 0$:(

$\theta_1: \frac{3\pi}{2} + 2\left(-\frac{\pi}{4}\right) = \pi$ ✓

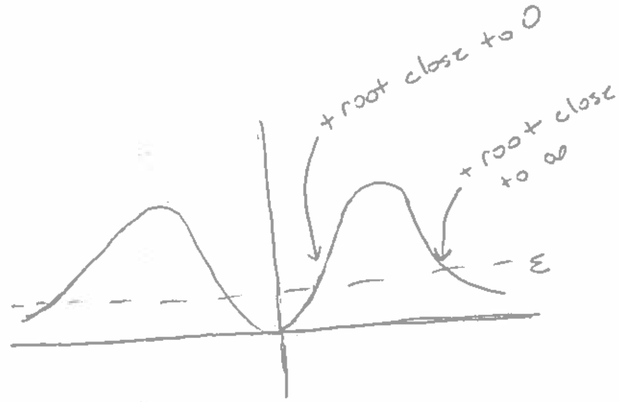
$\theta_1 = \pi$ is angle of SD

Problem 7

$$x^4 e^{-x^4} = \varepsilon \qquad (1)$$

$$x^4 = \varepsilon e^{x^4} \sim \varepsilon(1 + x^4) \qquad (2)$$

$$x_0 = \sum_{n=1}^{\infty} a_n \varepsilon^{n/4}$$



Note: Roots are all real

Sub into (2):

$$(a_1 \varepsilon^{1/4} + a_2 \varepsilon^{1/2} + a_3 \varepsilon^{3/4} + \dots)^4 = \varepsilon + \varepsilon(a_1 \varepsilon^{1/4} + a_2 \varepsilon^{1/2} + a_3 \varepsilon^{3/4} + \dots)^4$$

$O(\varepsilon)$: $a_1^4 = 1$

$a_1 = \cancel{\pm i}, \pm 1$ roots are real

$O(\varepsilon^{5/4})$: $4a_1^3 a_2 = 0$ $\quad a_2 = 0$

$O(\varepsilon^{3/2})$: $4a_1^3 a_3 = 0$ $\quad a_3 = 0$

$O(\varepsilon^{7/4})$: $4a_1^3 a_4 = 0$ $\quad a_4 = 0$

$O(\varepsilon^2)$: $4a_1^3 a_5 = a_1^4 \Rightarrow a_5 = \frac{a_1}{4} = \pm\frac{1}{4}$

$x_1 \sim \varepsilon^{1/4} + \frac{1}{4}\varepsilon^{5/4}$ pos root

$x_2 \sim -\varepsilon^{1/4} - \frac{1}{4}\varepsilon^{5/4}$ neg root

as $\varepsilon \to 0$

Pos root at ∞:

$$e^{-x^4} = \frac{\varepsilon}{x^4}$$

$$-x^4 \ln e = \ln\left(\frac{\varepsilon}{x^4}\right)$$

$$x^4 = \ln\left(\frac{x^4}{\varepsilon}\right)$$

no imaginary roots

$$x = \pm \sqrt[4]{\ln\left(\frac{x^4}{\varepsilon}\right)},\ \pm i\sqrt[4]{\ln\left(\frac{x^4}{\varepsilon}\right)}$$

Only want pos root

$$x = \sqrt[4]{\ln\left(\frac{x^4}{\varepsilon}\right)}$$

$$\Rightarrow \bar{x}_{n+1} = (4\ln \bar{x}_n - \ln\varepsilon)^{1/4}$$

$$\bar{x}_1 = (4\ln\bar{x}_0 - \ln\varepsilon)^{1/4} \qquad \text{Choose } \bar{x}_0 = \left(\ln\tfrac{1}{\varepsilon}\right)^{1/4}$$

$$\bar{x}_1 = \left(4\ln(\ln 1/\varepsilon)^{1/4} - \ln\varepsilon\right)^{1/4}$$

$$= (\ln(\ln 1/\varepsilon) - \ln\varepsilon)^{1/4}$$

$$x_3 \sim (\ln(\ln 1/\varepsilon) - \ln\varepsilon)^{1/4} \quad \text{as } \varepsilon \sim 0$$