METHODS OF APPLIED MATHEMATICS COMPREHENSIVE EXAMINATION AUGUST 2012

Work on as many of the following problems as possible. Turn in *all* your work.

(1) Apply iterated homogenization (average fastest scale first and repeat), on the following initial value problem to derive a leading order effective equation as $\epsilon \to 0^+$, assuming the functions $K(x), M(x)$ are periodic sharing the same period, and positive:

$$u_t = \left[\left(K(x/\epsilon) + M(x/\epsilon^2)\right) u_x\right]_x$$
$$u(x, t=0) = u_0(x)$$

(2) Consider the following "quarter-plane" boundary value problem for the diffusion equation with constant diffusivity κ,

$$c_t = \kappa\, c_{xx}\,, \qquad c(x,0) = 0\,, \qquad c(0,t) = t$$

(a) Solve for all time $t > 0$ in the half line $x \geq 0$.
(b) Discuss the limit $x \to 0^+$ (verify the boundary condition is satisfied).

(3) Consider the function $F(s;x)$ dependent on complex argument s and the real parameter $x > 0$,

$$F(s;x) = \frac{e^{-x\sqrt{s}}}{s^2}\,.$$

(a) Classify its singularities and define domains where it is single valued.
(b) Choose the domains in such a way that the integral depending on the parameter $c > 0$,

$$I(x,t) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{st} F(s;x)\, ds\,,$$

is well defined, and in particular discuss its convergence.
(c) Evaluate the integral along appropriate contours in the s-complex plane and discuss the first two terms of the asymptotic expansion of $I(x,t)$ as $t \to +\infty$.
(d) Discuss the limit $t \to 0^+$.

(4) Find a two term asymptotic expansion for all the (complex) roots of the following as $\epsilon \to 0$,

$$(x-1)^2 + \epsilon\sqrt{x} = 0\,.$$

(5) Compute the leading order term in an asymptotic expansion of the solution $y(x)$ of the following third order differential equation

$$y''' = xy + 2y$$

as $x \to \pm\infty$. Classify the different forms of the solutions with respect to their asymptotic behavior, and discuss their linear (in)dependency.

(6) (a) Compute the Green function $G(x, y)$ for the operator $\frac{d^2}{dx^2}$ on the interval $[0, 1]$, that is

$$\frac{\partial^2 G}{\partial x^2} = \delta(x - y), \qquad x \in [0, 1], \quad y \in [0, 1],$$

($\delta \equiv$ Dirac-delta) subject to the boundary conditions

$$G(0, y) = 0 = \frac{\partial G}{\partial x}(1, y).$$

(b) For a solution $\phi(x)$ of the (trivial) differential equation

$$\phi'' \equiv \frac{d^2\phi}{dx^2} = 0,$$

derive Green's identity

$$\phi(x) = \left[\phi(y)\,\frac{\partial G}{\partial y}(x, y) - \phi'(y)\,G(x, y)\right]_0^1$$

where the square brackets at the right hand side denote the difference $f(x, 1) - f(x, 0)$ for any function $f(x, y)$ of the arguments x, y.

(c) Hence compute the solution of the two-point boundary value problem on the interval $[0, 1]$,

$$\phi'' = 0, \qquad \phi'(0) = 1, \quad \phi(1) = 0.$$

How does the solution obtained with Green's identity compare with a direct solution of this two point boundary value problem?

(7) (a) Explain the difference between pointwise convergence and asymptotic convergence. Illustrate with the particular example of power series.

(b) Are asymptotic series unique? Explain

(c) Does an asympotic relation carry over to its integration? Discuss in the context of the following example: does

$$f(x) \sim g(x) \qquad \text{as} \quad x \to \infty$$

imply, for some x_0,

$$\int_{x_0}^{x} f(y)\,dy \sim \int_{x_0}^{x} g(y)\,dy \qquad \text{as} \quad x \to \infty?$$

Make sure to clarify your assumptions on f and x for a definitive, but non trivial, statement.

Problem 1

Aug 2012

$$u_t = \partial_x\Big(\underbrace{\big(K(\tfrac{x}{\varepsilon}) + M(\tfrac{x}{\varepsilon^2})\big)}_{L(x,\varepsilon)}\, u_x\Big)$$

$$u\big|_{t=0} = u_0(x)$$

Let $\omega = \frac{x}{\varepsilon^2}$, $z = \frac{x}{\varepsilon}$ $\Rightarrow$ $\partial_x \longmapsto \partial_x + \frac{1}{\varepsilon}\partial_z + \frac{1}{\varepsilon^2}\partial_\omega$

Ansatz: $u(x,\omega,z,t) = \bar{u}(x,\omega,z,t) + \varepsilon u_1(x,\omega,z,t) + \dots$

$$u_t = \left(\partial_x + \tfrac{1}{\varepsilon}\partial_z + \tfrac{1}{\varepsilon^2}\partial_\omega\right)\left[L(\omega,z)\left(\partial_x + \tfrac{1}{\varepsilon}\partial_z + \tfrac{1}{\varepsilon^2}\partial_\omega\right)u\right]$$

$\mathcal{O}(\frac{1}{\varepsilon^4})$: $\partial_\omega(L\partial_\omega \bar{u}) = 0 \Rightarrow \bar{u}(x,z,t)$

$\mathcal{O}(\frac{1}{\varepsilon^3})$: $\partial_\omega(L\partial_\omega u_1) + \cancel{\partial_z(L\partial_\omega \bar{u})}^{\,0} + \partial_\omega(L\partial_z\bar{u}) = 0$

$$\langle \partial_\omega(L\partial_\omega u_1 + L\partial_z\bar{u})\rangle_\omega = 0$$

$$L\partial_\omega u_1 + L\partial_z\bar{u} = A(x,z,t)$$

$$\langle \partial_\omega u_1\rangle_\omega + \langle\partial_z\bar{u}\rangle_\omega = \left\langle \frac{A}{L}\right\rangle_\omega$$

$$A = \langle L\rangle_\omega^h\, \partial_z\bar{u}$$

$$L(\partial_\omega u_1 + \partial_z\bar{u}) = \langle L\rangle_\omega^h\, \partial_z\bar{u}$$

$\mathcal{O}(\frac{1}{\varepsilon^2})$: $\partial_\omega(L\partial_\omega u_2) + \partial_z(L\partial_\omega u_1) + \partial_\omega(L\partial_z u_1) + \partial_z(L\partial_z\bar{u}) = 0$

$$\cancel{\langle \partial_\omega(L\partial_\omega u_2 + L\partial_z u_1)\rangle_\omega}^{\,0} + \langle\partial_z(L\partial_\omega u_1)\rangle_\omega + \langle\partial_z(L\partial_z\bar{u})\rangle = 0$$

$$\hookrightarrow L\partial_\omega u_2 + L\partial_z u_1 = \langle L\rangle_\omega^h\, \partial_z u_1$$

$$\Rightarrow u_1(x,z,t)$$

$$\Rightarrow \bar{u}(x,t)$$

$$\mathcal{O}(1/\varepsilon):\ \partial_w(L\partial_w u_3) + \partial_z(L\partial_w u_2) + \partial_w(L\partial_z u_2) + \partial_z(L\partial_z u_1) + \partial_z(L\partial_x \bar{u}) + \cancelto{0}{\partial_x(L\partial_z \bar{u})} = 0$$

$$\cancelto{0}{\langle \partial_w(L\partial_w u_3 + L\partial_z u_2)\rangle_w} + \langle \partial_z(L\partial_w u_2\rangle_w + \langle L\partial_z u_1 + L\partial_x \bar{u}))\rangle_w = 0$$

$$\cancelto{0}{\langle \partial_z \langle L\partial_w u_2\rangle_w\rangle_z} + \langle \langle L\rangle_w(\partial_z u_1 + \partial_x \bar{u})\rangle_z = 0$$

$$\Rightarrow \quad \langle L\rangle_w(\partial_z u_1 + \partial_x \bar{u}) = \langle\langle L\rangle_w\rangle_z^h\ \partial_x \bar{u}$$

$$\mathcal{O}(1):\ \partial_w(L\partial_w u_4) + \partial_z(L\partial_w u_3) + \partial_w(L\partial_z u_3) + \partial_z(L\partial_z u_2) + \partial_z(L\partial_x u_1) + \partial_x(L\partial_z u_1) + \partial_x(L\partial_x \bar{u}) = \bar{u}_t$$

Avg wrt w

$$\Rightarrow \partial_z[\langle L\rangle_w(\partial_z u_2 + \partial_x u_1)] + \partial_x[\underbrace{\langle L\rangle_w(\partial_z u_1 + \partial_x \bar{u})}_{\langle\langle L\rangle_w\rangle_z^h \bar{u}_x}] = \bar{u}_t$$

Avg wrt z

$$\langle \partial_z[\langle L\rangle_w(\partial_z u_2 + \partial_x u_1)]\rangle_z + \langle \partial_x[\langle\langle L\rangle_w\rangle_z^h\ \bar{u}_x]\rangle_z = \langle \bar{u}_t\rangle_z$$

$$\therefore \boxed{\bar{u}_t = \langle\langle L\rangle_w\rangle_z^h\ \bar{u}_{xx}}$$

Problem 2 (w/o const k) Aug 2012

$$\begin{cases} c_t = c_{xx} \\ c(x,0) = 0 \\ c(0,t) = t \end{cases}$$

$$\widehat{c_t} = \int_0^\infty e^{-st} \frac{\partial}{\partial t} c(x,t)\,dt = \cancel{e^{-st} c(t,x)\Big|_0^\infty}^{\to 0} + s\int_0^\infty e^{-st} c(t,x)\,dx = s\hat{c}$$

$$\widehat{c_{xx}} = \hat{c}_{xx}$$

$$s\hat{c} = \hat{c}_{xx}$$

$$\hat{c}(s,x) = A(s)\, e^{-x\sqrt{s}} + B(s)\, e^{x\sqrt{s}}$$

For physically rellavent soln set $B(s) = 0$ assuming we pick + branch of $\sqrt{\ }$.

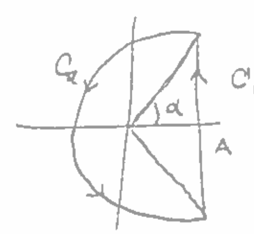
$$\hat{c}(0,s) = \int_0^\infty e^{-st}\, t\,dt = t\left(-\tfrac{1}{s}\right)e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt = \frac{1}{s^2} = A(s$$

$$\hat{c}(x,s) = \frac{1}{s^2} e^{-x\sqrt{s}} \Rightarrow \boxed{c(x,t) = \frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty} \frac{e^{st - x\sqrt{s}}}{s^2}\,ds}$$

b) By Dominant Conv Thm

$$\lim_{x\to 0^+} c(x,t) = \frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty} \lim_{x\to 0^+} \frac{e^{st} e^{-x\sqrt{s}}}{s^2}\,ds = \frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty} \frac{e^{st}}{s^2}\,ds$$

$\tilde{C}$:
$$\frac{1}{2\pi i}\int_{\tilde{C}} \frac{e^{st}}{s^2}\,ds = \frac{2\pi i}{2\pi i}\,\mathrm{Res}\left(\frac{e^{st}}{s^2}, s=0\right)$$
$$= \lim_{s\to 0} \frac{d}{ds} e^{st} = \lim_{s\to 0}(te^{st}) = t$$



$\tilde{C} = C_1 + C_2$

Note this means $\rho\cos\theta \le A$ for $\theta \in (\alpha, 2\pi - \alpha)$

C_2: $s = \rho e^{i\theta} \Rightarrow ds = i\rho e^{i\theta}\,d\theta \quad \theta \in (\alpha, 2\pi-\alpha)$

$$\left|\int_{C_2} \frac{e^{st}}{s^2}\,ds\right| = \left|\int_\alpha^{2\pi-\alpha} \frac{e^{\rho e^{i\theta} t}}{\rho^2 e^{2i\theta}}\,(i\rho e^{i\theta})\,d\theta\right| \le \int_\alpha^{2\pi-\alpha} \frac{|e^{\rho e^{i\theta} t}|}{\rho\,|e^{2i\theta}|}\,d\theta$$

$$\le \int_\alpha^{2\pi-\alpha} \frac{e^{At}}{\rho}\,d\theta \to 0 \quad \text{as } \rho \to \infty$$

By Cauchy's Thm

$$\frac{1}{2\pi i}\int_{\tilde{C}} \frac{e^{st}}{s^2}\,ds = \frac{1}{2\pi i}\int_{C_1} \frac{e^{st}}{s^2}\,ds + \frac{1}{2\pi i}\int_{C_2} \frac{e^{st}}{s^2}\,ds$$

$$\frac{1}{2\pi i}\int_{C_1} \frac{e^{st}}{s^2}\,ds = \frac{1}{2\pi i}\int_{C} \frac{e^{st}}{s^2}\,ds - \frac{1}{2\pi i}\int_{C_2} \frac{e^{st}}{s^2}\,ds \quad = t - 0 = t$$

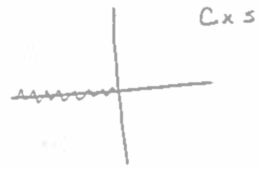
Problem 3

$$F(s,x) = \frac{e^{-x\sqrt{s}}}{s^2}$$

a) $g(s) = \sqrt{s}$ $\quad s = 0 \quad$ branch pt

$h(s) = s^2$ $\quad s = 0 \quad$ double pole

Choose branch cut:



← Domain where $F(s,x)$ is sing. val.

b) $$I(x,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st - x\sqrt{s}}}{s^2}\, ds$$

$\frac{1}{2\pi i}$ will ignore for now

change of var: $s = c + iy$

$ds = i\,dy$

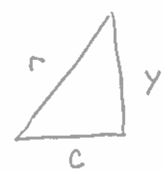
$s = re^{i\theta}$ $\quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\because c > 0$

$$|I(x,t)| \le \left| \int_{c-i\infty}^{c+i\infty} \frac{e^{st - x\sqrt{s}}}{s^2}\, ds \right|$$

$$= \left| \int_{-\infty}^{\infty} \frac{e^{(c+iy)t - x\sqrt{r}e^{i\frac{\theta}{2}}}}{(re^{i\theta})^2}\, i\,dy \right|$$

$$\le \int_{-\infty}^{\infty} \frac{|e^{iyt}|\,|e^{ct - x\sqrt{r}\cos\frac{\theta}{2}}|\,|e^{ix\sqrt{r}\sin\frac{\theta}{2}}|}{|r|^2}\, dy$$



$r = \sqrt{c^2 + y^2}$

$$\le \int_{-\infty}^{\infty} \frac{e^{ct - x(c^2+y^2)^{1/4}\cos\frac{\theta}{2}}}{|c^2 + y^2|}\, dy$$

$$\le \frac{e^{ct}}{c^2} \int_{-\infty}^{\infty} e^{-x(c^2+y^2)^{1/4} \overbrace{\cos\frac{\theta}{2}}^{>0}}$$

$< \infty$ $\quad$ for $\quad x > 0$

The domain where $I(x,t)$ is well-defined and convergent is: $\{(x,t) : x > 0,\ t \in \mathbb{R}\}$

Note: If more than a simple pole, int. by parts

c) $I(x,t) \cong \int_{c-i\infty}^{c+i\infty} \frac{e^{st-x\sqrt{s}}}{s^2}\,ds \sim ?$ as $t \to \infty$

$u = e^{st-x\sqrt{s}}$ $\quad dv = \frac{1}{s^2}\,ds$

$du = \left(t - \frac{x}{\sqrt{s}}\right) e^{st-x\sqrt{s}}\,ds \quad v = -\frac{1}{s}$

$$= \underbrace{-\frac{e^{st-x\sqrt{s}}}{s}\Bigg|_{c-i\infty}^{c+i\infty}}_{\text{Part 1}} + \underbrace{\int_{c-i\infty}^{c+i\infty} \left(t - \frac{x}{2\sqrt{s}}\right)\frac{e^{st-x\sqrt{s}}}{s}\,ds}_{\text{Part 2}}$$

Part 1: $s = c+iy = re^{i\theta}$ $\quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\Rightarrow \left|-\frac{e^{st-x\sqrt{s}}}{s}\right| = \left|\frac{e^{(c+iy)t - x\sqrt{r}e^{i\theta/2}}}{c+iy}\right| \leq \frac{e^{ct}}{c}\left|e^{-x(c^2+y^2)^{1/4}\overbrace{\cos\frac{\theta}{2}}^{>0}}\right| \longrightarrow 0 \text{ as } y \to \pm\infty$$

Part 2:

$$\underbrace{t\int_{c-i\infty}^{c+i\infty} \frac{e^{st-x\sqrt{s}}}{s}\,ds}_{①} - \underbrace{\frac{x}{2}\int_{c-i\infty}^{c+i\infty} \frac{e^{st-x\sqrt{s}}}{s\sqrt{s}}\,ds}_{②}$$

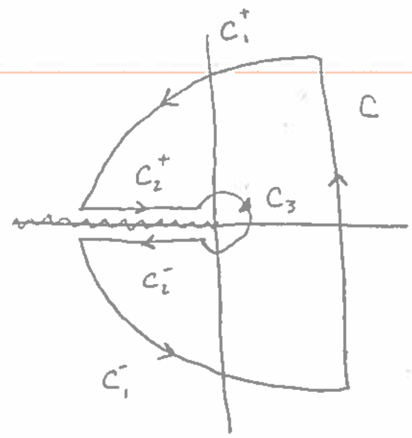
$$|②| \leq \frac{x}{2\sqrt{c}}\left|\int_{c-i\infty}^{c+i\infty} \frac{e^{st-x\sqrt{s}}}{s}\,ds\right| \underset{\substack{\uparrow \\ \text{as } t\to\infty}}{\lessapprox} \frac{x/2\sqrt{c}}{\left|t\int_{c-i\infty}^{c+i\infty}\frac{e^{st-x\sqrt{s}}}{s}\,ds\right|} \cdot |①|$$

$\Rightarrow$ ② $\ll$ ① as $t \to \infty$

$$\therefore I(x,t) \sim \frac{t}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st-x\sqrt{s}}}{s}\,ds \quad \text{as } t \to \infty$$

Let $\tilde{C} = C + C_1^+ + C_1^- + C_2^+ + C_2^- + C_3$

$$\Rightarrow t\int_{\tilde{C}} \frac{e^{st-x\sqrt{s}}}{s} = 0$$



$\underline{C_1^+}$: $s = re^{i\theta}$ $\quad \theta \in (\alpha, \pi)$

$\theta \in (\alpha, \pi/2) \Rightarrow r\cos\theta < c$

$\theta \in (\frac{\pi}{2}, \pi) \Rightarrow \cos\theta < 0$

$ds = ire^{i\theta}\,d\theta$

Note: $\frac{\theta}{2} \in (\frac{\alpha}{2}, \frac{\pi}{2})$
$\Rightarrow \cos\theta/2 > 0$

$$\left|\int_{C_1} \frac{e^{st - x\sqrt{s}}}{s}\,ds\right| \le \left|\int_\alpha^{\pi/2} \frac{|e^{\overbrace{r\cos\theta}^{<c}\,t - x\sqrt{r}\cos\frac{\theta}{2}}|}{\cancel{|r|}} \cancel{|r}e^{i\theta}|\,d\theta\right|$$

$$+ \left|\int_{\pi/2}^{\pi} \frac{|e^{\overbrace{r\cos\theta}^{<0}\,t - x\sqrt{r}\cos\frac{\theta}{2}}|}{\cancel{|r|}} \cancel{|r}e^{i\theta}|\,d\theta\right|$$

$$\le \int_\alpha^{\pi/2} e^{ct} e^{-x\sqrt{r}\cos\frac{\theta}{2}}\,d\theta \quad (\longrightarrow 0 \text{ as } r \to \infty)$$

$$+ \int_{\pi/2}^{2} e^{\underbrace{r\cos\theta t - x\sqrt{r}\cos\frac{\theta}{2}}_{<0}}\,d\theta \quad (\longrightarrow 0 \text{ as } r \to \infty)$$

$$\longrightarrow 0 \text{ as } r \to \infty$$

$\underline{C_1^-}$: Simillar to C_1^+ but $\theta \in (-\alpha, -\pi)$

$\theta \in (-\alpha, -\frac{\pi}{2}) \Rightarrow r\cos\theta < c$

$\theta \in (-\frac{\pi}{2}, -\pi) \Rightarrow \cos\theta < c$

Contribution will also be 0 as $r \to \infty$

$\underline{C_3}$: $$t\int_{C_3} \frac{e^{st - x\sqrt{s}}}{s}\,ds = \int_\pi^{-\pi} \frac{e^{\varepsilon e^{i\theta} t - x\sqrt{\varepsilon}e^{i\theta/2}}}{\cancel{\varepsilon e^{i\theta}}} i\cancel{\varepsilon e^{i\theta}}\,d\theta \longrightarrow -2\pi i t \text{ as } \varepsilon \to 0$$

$s = \varepsilon e^{i\theta}$

$ds = i\varepsilon e^{i\theta}\,d\theta$

$\underline{C_2^+:}$ $\quad s = r e^{i\pi} \qquad ds = e^{i\pi} dr$

$$t \int_{\infty}^{0} \cancel{e^{i\pi}} \frac{e^{r e^{i\pi} t - x\sqrt{r} e^{i\pi/2}}}{r \cancel{e^{i\pi}}} dr = -t \int_0^{\infty} \frac{e^{-rt - ix\sqrt{r}}}{r} dr$$

$\underline{C_2^-:}$ $\quad s = r e^{-i\pi} \qquad ds = e^{-i\pi} dr$

$$t \int_0^{\infty} e^{-i\pi} \frac{e^{r e^{i\pi} t - x\sqrt{r} e^{-i\pi/2}}}{r} dr = t \int_0^{\infty} \frac{e^{-rt + ix\sqrt{r}}}{r} dr$$

$$C_2^+ + C_2^- = t \int_0^{\infty} \frac{e^{-rt}}{r} \left(e^{ix\sqrt{r}} + e^{-ix\sqrt{r}}\right) dr$$

$$= 2\pi i t \int_0^{\infty} \frac{e^{-rt}}{r} \left(\sin(x\sqrt{r})\right) dr$$

$\underline{C:}$ $\quad \int_C = -\int_{C_3} - \left(\int_{C_2^+} + \int_{C_2^-}\right) \quad$ if $r \to \infty$ and $\varepsilon \to 0$

$$t \int_{c-i\infty}^{c+i\infty} \frac{e^{st - x\sqrt{s}}}{s} ds = 2\pi i t - 2\pi i t \int_0^{\infty} \underbrace{\frac{\sin(x\sqrt{r})}{r}}_{f(r)} e^{-rt} dr$$

Use Watson's Lemma

$$\sin(x\sqrt{r}) \sim x\sqrt{r} - \frac{x^3 r^{3/2}}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} r^{(2n+1)/2}$$

$$\frac{\sin(x\sqrt{r})}{r} \sim r^{-1/2} \sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{a_n} r^n$$

$\alpha = -\frac{1}{2} \qquad \beta = 1$

$$\int_0^{\infty} \frac{\sin(x\sqrt{r})}{r} e^{-rt} dr \sim \frac{x\,\Gamma\left(-\frac{1}{2} + 1(0) + 1\right)}{t^{\left(-\frac{1}{2} + 1(0) + 1\right)}} \to \frac{x\,\overset{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}\right)}}{\sqrt{t}}$$

by Watson's Lemma

$$I(x,t) \sim \frac{1}{2\pi i}\left(2\pi i t - 2\pi i t\left(\frac{x\sqrt{\pi}}{\sqrt{t}}\right)\right) \quad \text{as } t \to \infty$$

$$\boxed{I(x,t) \sim t - x\sqrt{\pi t} \quad \text{as } t \to \infty}$$

d) The only place we used $t \to \infty$ was when we showed ② << ①. But if $t \to 0^+$, then ① << ②. This means our integral should be multiplied by $-\frac{x}{4\pi i}$ instead of $\frac{t}{2\pi i}$. All of the work done for the contour integration stays the same.

Thus:

$$I(x,t) \sim -\frac{x}{2t}\left(t - x\sqrt{\pi t}\right) \quad \text{as } t \to 0^+$$

$$\boxed{I(x,t) \sim \frac{x\sqrt{\pi}}{\sqrt{t}} - \frac{x}{2} \quad \text{as } t \to 0}$$

Problem 4

Aug 2012

$$(x-1)^2 + \varepsilon\sqrt{x} = 0 \qquad (1)$$

$\varepsilon = 0 \Rightarrow \quad (x-1)^2 = 0$

$\qquad x = 1, 1$

$x_{1,2} \sim 1 + a\varepsilon^{\alpha}$

Sub into (1):

$$(1 + a\varepsilon^{\alpha} - 1)^2 + \varepsilon(1 + a\varepsilon^{\alpha})^{\frac{1}{2}} = 0$$

$$\varepsilon(1 + a\varepsilon^{\alpha})^{\frac{1}{2}} = -a\varepsilon^{\alpha} \qquad *$$

$$\varepsilon^2(1 + a\varepsilon^{\alpha}) = a^2\varepsilon^{2\alpha}$$

$$\varepsilon^2 + a\varepsilon^{\alpha+2} - a^2\varepsilon^{2\alpha} = 0$$

$2\alpha = 2$

$\alpha = 1 \quad \Rightarrow \quad$ consistent

$2\alpha = \alpha + 2$

$\alpha = 2 \quad \Rightarrow \quad$ inconsistent

$2 = \alpha + 2$

$\alpha = -1 \quad \Rightarrow \quad \varepsilon^{\alpha} \gg 1$:(

$$\varepsilon^2 + a\varepsilon^3 - a^2\varepsilon^2 = 0$$

$O(\varepsilon^2)$: $\quad 1 - a^2 = 0$

$\qquad a = \pm 1$

$$\boxed{x_{1,2} \sim 1 \pm \varepsilon}$$