# METHODS OF APPLIED MATHEMATICS COMPREHENSIVE EXAMINATION AUGUST 2016

Work on as many of the following problems as possible. Turn in *all* your work.

(1) Consider two bodies of mass $m_1$ and $m_2$, respectively, joined by a spring (Hooke's law, constant $k$), in collinear motion along the $z$-axis of a cartesian frame in $\mathbb{R}^3$. If the position of the first body is at $z = r_1(t)$ and the second is at $z = r_2(t)$, and each body body is also subject to an overall central force field $F = -Km_j/z^4$, where $z \neq 0$, $m_1, m_2$ are the body's masses and $K$ is a constant, then
  (a) Write evolution equations given by Newton's second law for the functions $r_1$ and $r_2$ of time $t$.
  (b) Identify the units of $K$.
  (c) Non-dimensionalize the equations of motion; identify non-dimensional parameters; discuss all the possible dominant balances.
  (d) Neglecting the central force field, find the solution of these equations corresponding to zero initial velocities, taking $r_1(0) - r_2(0) = h > 0$.
  (e) Write an asymptotic expansion for the leading order plus the first correction of the solution assuming $h \ll (r_1(0) + r_2(0))/2$, and define the range of initial conditions that allow the central force to be considered weaker than the spring force.
  (f) Sketch the leading order plus the first correction of the solutions and note their time scales of validity.

(2) Consider the integral

$$I(a) = \int_0^{2\pi} \frac{dx}{a + i\cos x} \qquad a \in \mathbb{R}$$

  (a) Prove that $I(a)$ is real.
  (b) Show that $I(a)$ is and odd function of $a$, i.e., $I(-a) = -I(a)$
  (c) Continue the integrand in the complex plane $z$, $z = x + iy$, (i.e., $\Re(z) = x$), and use the residue theorem to show that for $a > 0$

$$I(a) = \frac{2\pi}{\sqrt{1 + a^2}}$$

  This result seemingly suggests that $I(a)$ is an even function of $a$. How can this be reconciled with the odd property of $I(a)$ proved above? Discuss.

(3) Consider the eigenvalue problem on the real line $x \in \mathbb{R}$

$$\epsilon y'' - (U(x) + \lambda)y = 0, \qquad y(x) \to 0, \text{ as } |x| \to \infty.$$

  with the (square well) potential $U(x) = 1$ for $|x| > 1$ and $U(x) = -2$ for $|x| < 1$
  (a) Identify the range of $\lambda$ for bounded eigenfunctions to exist.
  (b) Solve for eigenvalues and eigenfunctions exactly.
  (c) Study their asymptotic limit as $\epsilon \to 0$.
  (d) As $\epsilon \to 0$ compute acceptable solutions directly via WKBJ approach using the two-turning point analysis in this limit. Compare the result with the exact formulae.

(4) Consider the rapidly varying diffusivity:

$$K(x,y,z;\epsilon) = A + F(x/\epsilon^3) + G(y/\epsilon^2) + H(z/\epsilon)$$

where $A$ is chosen to guarantee $K$ is positive, and $\epsilon$ is a small constant. By applying iterated homogenization, average the following diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K(x,y,z;\epsilon)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K(x,y,z;\epsilon)\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(K(x,y,z;\epsilon)\frac{\partial u}{\partial z}\right)$$
$$u(x,y,0) = u_0(x,y,z)\,,$$

by computing a leading order effective equation governing the evolution as $\epsilon \to 0$ over the $(x,y)$-plane, assuming the functions $F(x)$, $G(y)$ and $H(z)$ are mean zero, periodic, and share the same period. Solve the averaged equation in free space.

(5) Consider the following functions:

$$f(x,t) = \frac{2xt[(1-t^2)^2 - x^2]}{(1-t^2)^{3/2}[t^2(1-t^2)^2 + x^2]}$$
$$g(x,t) = \frac{2xt}{t^2(1-t^2)^2 + x^2}$$

(a) Is the function $g$ asymptotic to $f$ in an interval, $-t^\beta < x < t^\delta$, as $t \to 0^+$? If so, for what values of the parameters $\beta$ and $\delta$? Discuss.
(b) Estimate the size of the extrema of $f$ and their $x$-locations and as $t \to 0^+$.

(6) Consider the following initial value problem for the time-$t$ evolution equation in one spatial dimension $x \in \mathbb{R}$

$$T_t + \gamma\,(2+\cos t)x\,T_x = \kappa T_{xx}\,, \qquad T(x,0) = T_0(\alpha x)\,.$$

(a) What are the units of the parameters $\gamma$, $\kappa$ and $\alpha$?
(b) Non-dimensionalize the equation.
(c) Use the Fourier-transform method with definition

$$\hat{T}(k,t) \equiv \int_{-\infty}^{\infty} T(x,t)e^{-ikx}\,dx$$

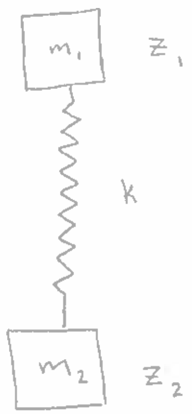
to solve the resulting equation for $\hat{T}$ by the method of characteristics.
(d) What PDE does $u \equiv T_x$ solve?
(e) Compare the long-time asymptotics of $T$ vs. $u$ assuming that $T_0$ is a Heaviside step function.

(7) (a) Explain the difference between pointwise convergence and asymptotic convergence. Illustrate with the particular example of power series.
(b) Define uniform asymptotic convergence.
(c) Are the functions $f$ and $g$ in problem (5) uniformly asymptotic to each other as $t \to 0^+$ in some $t$-dependent interval containing the origin?
(d) Are the derivatives, $\frac{\partial f}{\partial t}$ and $\frac{\partial g}{\partial t}$ uniformly asymptotic to each other as $t \to 0^+$ on some $t$-dependent interval containing the origin? If so, estimate the interval size. What about the partial $x$-derivatives?

Aug 2011

# Problem 1

a)




$$\begin{cases} m_1 \ddot{z}_1 = -k(z_1 - z_2 - d) - \dfrac{K m_1}{z_1^4} \\ m_2 \ddot{z}_2 = k(z_1 - z_2 - d) - \dfrac{K m_2}{z_2^4} \end{cases}$$

$z_1(0) = \bar{z}_1$

$z_2(0) = \bar{z}_2$

$d =$ length of spring at rest

b) $[m \ddot{z}] = \dfrac{kg \cdot m}{s^2}$

$\left[\dfrac{m}{z^4}\right] = \dfrac{kg}{m^4}$

$[K] = \dfrac{kg \cdot m}{s^2} \cdot \dfrac{m^4}{kg} = \boxed{\dfrac{m^5}{s^2} = [K]}$

c) Let $L = d$ and $T = \sqrt{\dfrac{m_1}{k}}$ then

$z = Lx$ and $t = T\tau$

non-dimensional

$$\frac{dz}{dt} = L\frac{dx}{dt} = \frac{L}{T}\frac{dx}{d\tau} = \frac{L}{T}\dot{x}$$

$$\frac{d^2z}{dt^2} = \frac{d}{dt}\left(\frac{L}{T}\frac{dx}{d\tau}\right) = \frac{L}{T^2}\ddot{x}$$

$$\begin{cases} m_1 \dfrac{L}{T^2}\ddot{x}_1 = -k(Lx_1 - Lx_2 - L) - \dfrac{K m_1}{L^4 x_1^4} \\ m_2 \dfrac{L}{T^2}\ddot{x}_2 = k(Lx_1 - Lx_2 - L) - \dfrac{K m_2}{L^4 x_2^4} \end{cases}$$

$$\boxed{\begin{cases} \ddot{x}_1 = -\dfrac{kT^2}{m_1}(x_1 - x_2 - 1) - \dfrac{KT^2}{L^5 x_1^4} \\ \ddot{x}_2 = \dfrac{kT^2}{m_2}(x_1 - x_2 - 1) - \dfrac{KT^2}{L^5 x_2^4} \end{cases}}$$

# Problem 2

Aug 2016

$$I(a) = \int_0^{2\pi} \frac{dx}{a + i\cos x} \qquad a \in \mathbb{R}$$

a) $$I(a) = \int_0^{2\pi} \frac{dx}{a + i\cos x} = \int_0^{2\pi} \frac{a - i\cos x}{a^2 + \cos^2 x}\, dx$$

$$= \underbrace{\int_0^{2\pi} \frac{a}{a^2+\cos^2 x}\, dx}_{\in \mathbb{R}} - i \underbrace{\int_0^{2\pi} \frac{\cos x}{a^2+\cos^2 x}\, dx}_{\text{want } = 0}$$

$$\int_0^{2\pi} \frac{\cos x}{a^2+\cos^2 x}\, dx = \int_0^{\pi} \frac{\cos x}{a^2+\cos^2 x}\, dx + \int_{\pi}^{2\pi} \frac{\cos x}{a^2+\cos^2 x}\, dx$$

Let $\tilde{x} = x - \pi \Rightarrow x = \tilde{x} + \pi$

$dx = d\tilde{x}$

$$\cos(x) = \cos(\tilde{x} + \pi) = -\cos\tilde{x}$$

$$\int_{\pi}^{2\pi} \frac{\cos x}{a^2+\cos^2 x}\, dx = \int_0^{\pi} \frac{-\cos\tilde{x}}{a^2+\cos^2\tilde{x}}\, d\tilde{x}$$

$$\Rightarrow \int_0^{2\pi} \frac{\cos x}{a^2+\cos^2 x}\, dx = \int_0^{\pi} \frac{\cos x}{a^2+\cos^2 x}\, dx - \int_0^{\pi} \frac{\cos\tilde{x}}{a^2+\cos^2\tilde{x}}\, d\tilde{x} = 0$$

b) $$I(-a) \underset{\text{by (a)}}{=} \int_0^{2\pi} \frac{(-a)}{(-a)^2+\cos^2 x}\, dx = -\int_0^{2\pi} \frac{a}{a^2+\cos^2 x}\, dx = -I(a)$$

$\therefore I(a)$ is an odd func

c) Let $z = e^{ix}$ $\Rightarrow$ $dz = ie^{ix}dx$ $\Rightarrow$ $dx = -i\frac{dz}{z}$

Note: $\cos x = \frac{z + \frac{1}{z}}{2}$

$$I(a) = \int_0^{2\pi} \frac{dx}{a + i\cos x} = \oint_c \frac{-i\,dz}{z(a + \frac{i}{2}(z + 1/z))} = -2i\oint_c \frac{dz}{-2za + iz^2 + i}$$

$$= -2\oint_c \frac{dz}{z^2 - 2aiz + 1} \qquad z_1 = \frac{2a \pm \sqrt{-4a^2 - 4}}{2} = a \pm i\sqrt{a^2+1} = z_{1,2}$$

$$= -2\oint_c \frac{dz}{(z - (a + i\sqrt{a^2+1}))(z - (a - i\sqrt{a^2+1}))} = 2\oint_c \frac{dz}{(z - z_1)(z - z_2)}$$

Since $a > 0$ $z_2 = a - \sqrt{a^2+1}$ is inside the unit circle so:

$$I(a) = -2(2\pi i)\,\mathrm{Res}\left(\frac{1}{(z - z_1)(z - z_2)},\ z = z_2\right)$$

$$= \frac{-4\pi i}{z_2 - z_1} = \frac{-4\pi i}{\cancel{a} - i\sqrt{a^2+1} - (\cancel{a} + i\sqrt{a^2+1})}$$

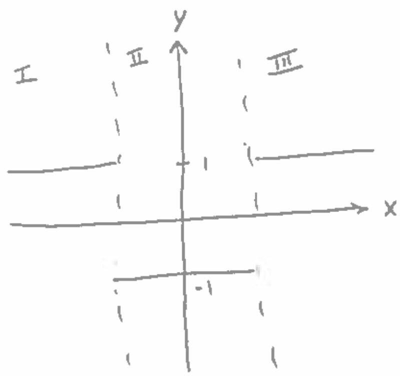
$$= \frac{-4\pi i}{-2i\sqrt{a^2+1}} = \frac{2\pi}{\sqrt{a^2+1}}$$

The function $f(x) = x$ is an odd function but the func $g(x) = e^{ix}$ is not. This change of variable is probably what introduced the discrepancy. Because of the change of variable we also introduced a sqrt of a sqr which can mess up the oddness of a function.

# Problem 3

$$\varepsilon y'' - (\mathcal{U}(x) + \lambda)\, y = 0 \qquad \Rightarrow \qquad y'' = \frac{\mathcal{U}(x) + \lambda}{\varepsilon}\, y$$

$$\mathcal{U}(x) = \begin{cases} 1 & |x| > 1 \\ -2 & |x| < 1 \end{cases}$$




a) $\max(\mathcal{U}(x)) > E > \min(\mathcal{U}(x))$

$$1 > -\lambda > -2$$

$$-1 < \lambda < 2$$

b) Region I: $\mathcal{U}(x) = 1$

$$y_1'' = \frac{1+\lambda}{\varepsilon}\, y_1 \qquad \text{Note: } \frac{1+\lambda}{\varepsilon} > 0$$

$$\Rightarrow \quad y_1 = A_1 e^{\sqrt{\frac{1+\lambda}{\varepsilon}}\, x} + B_1 e^{-\sqrt{\frac{1+\lambda}{x}}\, x}$$

But $y_1 \to 0$ as $x \to -\infty$ so $B_1 = 0$

$$\Rightarrow \quad y_1 = A_1 e^{\sqrt{\frac{1+\lambda}{\varepsilon}}\, x}$$

Region II: $\mathcal{U}(x) = -2$ $\qquad$ Note: $\frac{-2+\lambda}{\varepsilon} < 0$

$$\Rightarrow \quad y_2 = A_2 e^{\sqrt{\frac{-2+\lambda}{\varepsilon}}\, x} + B_2 e^{-\sqrt{\frac{-2+\lambda}{\varepsilon}}}$$

Region III: $u(x) = 1$ Note: $\frac{1+\lambda}{\varepsilon} > 0$

$$y_3 = A_3 e^{\sqrt{\frac{1+\lambda}{\varepsilon}}x} + B_3 e^{-\sqrt{\frac{1+\lambda}{\varepsilon}}x}$$

But $y_3 \to 0$ as $x \to \infty$ so $A_3 = 0$

$$\Rightarrow y_3 = B_3 e^{-\sqrt{\frac{1+\lambda}{\varepsilon}}x}$$

Patching:

$y_1(-1) = y_2(-1)$

$$A_1 e^{-\sqrt{\frac{1+\lambda}{\varepsilon}}} = -A_2 \sin\left(\sqrt{\frac{-2+\lambda}{\varepsilon}}\right) + B_2 \cos\left(\sqrt{\frac{-2+\lambda}{\varepsilon}}\right)$$

$y_1'(-1) = y_2'(-1)$

$$A_1\sqrt{1+\lambda}\, e^{-\sqrt{\frac{1+\lambda}{\varepsilon}}} = A_2\sqrt{-2+\lambda}\cos\left(\sqrt{\frac{-2+\lambda}{\varepsilon}}\right) + B_3\sqrt{-2+\lambda}\sin\left(\sqrt{\frac{-2+\lambda}{\varepsilon}}\right)$$

$y_2(1) = y_3(1)$

$$A_2 \sin\left(\sqrt{\frac{-2+\lambda}{\varepsilon}}\right) + B_2 \cos\left(\sqrt{\frac{-2+\lambda}{\varepsilon}}\right) = B_3 e^{-\sqrt{\frac{1+\lambda}{\varepsilon}}}$$

$y_2'(1) = y_3'(1)$

$$A_2\sqrt{-2+\lambda}\cos\left(\sqrt{\frac{-2+\lambda}{\varepsilon}}\right) - B_2\sqrt{-2+\lambda}\sin\left(\sqrt{\frac{-2+\lambda}{\varepsilon}}\right) = -B_3\sqrt{1+\lambda}\, e^{-\sqrt{\frac{1+\lambda}{\varepsilon}}}$$

$$\underbrace{\begin{pmatrix} e^{-\sqrt{\frac{1+\lambda}{\varepsilon}}} & \sin\left(\sqrt{\frac{-2+\lambda}{\varepsilon}}\right) & -\cos\left(\sqrt{\frac{-2+\lambda}{\varepsilon}}\right) & 0 \\ \sqrt{1+\lambda}\, e^{-\sqrt{\frac{1+\lambda}{\varepsilon}}} & -\sqrt{-2+\lambda}\cos\left(\sqrt{\frac{-2+\lambda}{\varepsilon}}\right) & -\sqrt{-2+\lambda}\sin\left(\sqrt{\frac{-2+\lambda}{\varepsilon}}\right) & 0 \\ 0 & \sin\sqrt{\frac{-2+\lambda}{\varepsilon}} & \cos\sqrt{\frac{-2+\lambda}{\varepsilon}} & e^{-\sqrt{\frac{1+\lambda}{\varepsilon}}} \\ 0 & \sqrt{-2+\lambda}\cos\left(\sqrt{\frac{-2+\lambda}{\varepsilon}}\right) & \sqrt{-2+\lambda}\sin\left(\sqrt{\frac{-2+\lambda}{\varepsilon}}\right) & \sqrt{1+\lambda}\, e^{-\sqrt{\frac{1+\lambda}{\varepsilon}}} \end{pmatrix}}_{M} \begin{pmatrix} A_1 \\ A_2 \\ B_2 \\ B_3 \end{pmatrix} = \vec{0}$$

We want $\det(M) = 0$ to ensure we don't have trivial solutions

<u>To finish</u>:

1) Take det. of $M$
2) Factor result
3) set each factor equal to zero
4) solve each resulting eqn for $\lambda$

c) To do this, I would need to finish part b.

But the result here should be equivalent to

what is found in part d.

d) $\frac{1}{\sqrt{\varepsilon}} \int_A^B \sqrt{E - Q}\, dx = \frac{1}{\sqrt{\varepsilon}} \int_{-1}^{1} \sqrt{-\lambda - (-2)}\, dx \sim (n + \tfrac{1}{2})\pi + O(\varepsilon)$ as $\varepsilon \to 0$ for $n \in \mathbb{Z}$

$$\frac{1}{\sqrt{\varepsilon}} \sqrt{2-\lambda} \int_{-1}^{1} dx = (n + \tfrac{1}{2})\pi$$

$$\frac{2}{\sqrt{\varepsilon}} \sqrt{2-\lambda} = (n + \tfrac{1}{2})\pi$$

$$2 - \lambda = \frac{\varepsilon (n + \tfrac{1}{2})^2 \pi^2}{4}$$

$$\boxed{\lambda = 2 - \frac{\varepsilon (n + \tfrac{1}{2})^2 \pi^2}{4}}$$

## Problem 4

Aug 2016

$$u_t = \partial_x(K\partial_x u) + \partial_y(K\partial_y u) + \partial_z(K\partial_z u)$$

$$u(x,y,z,0) = u_0(x,y,z)$$

$$K(x,y,z,\varepsilon) = A + F\left(\tfrac{x}{\varepsilon^3}\right) + G\left(\tfrac{y}{\varepsilon^2}\right) + H\left(\tfrac{z}{\varepsilon}\right)$$

Let $a = \frac{x}{\varepsilon^3}$ $\quad b = \frac{y}{\varepsilon^2}$ $\quad c = \frac{z}{\varepsilon}$

$$\Rightarrow \quad \partial_x \mapsto \partial_x + \tfrac{1}{\varepsilon^3}\partial_a$$

$$\partial_y \mapsto \partial_y + \tfrac{1}{\varepsilon^2}\partial_b$$

$$\partial_z \mapsto \partial_z + \tfrac{1}{\varepsilon}\partial_c$$

Ansautz: $\quad u = \bar{u} + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$

$$u_t = \left(\partial_x + \tfrac{1}{\varepsilon^3}\partial_a\right)\left(K\left(\partial_x + \tfrac{1}{\varepsilon^3}\partial_a\right)u\right) + \left(\partial_y + \tfrac{1}{\varepsilon^2}\partial_b\right)\left(K\left(\partial_y + \tfrac{1}{\varepsilon^2}\partial_b\right)u\right)$$

$$+ \left(\partial_z + \tfrac{1}{\varepsilon}\partial_c\right)\left(K\left(\partial_z + \tfrac{1}{\varepsilon}\partial_c\right)u\right)$$

$\mathcal{O}\left(\frac{1}{\varepsilon^6}\right)$: $\partial_a(K\partial_a \bar{u}) = 0$

$\Rightarrow$ $\bar{u}$ is indep of $a$

$\mathcal{O}\left(\frac{1}{\varepsilon^5}\right)$: $\partial_a(K\partial_a u_1) = 0 \quad \Rightarrow \quad u_1$ indep $a$

$\mathcal{O}\left(\frac{1}{\varepsilon^4}\right)$: $\partial_a(K\partial_a u_2) + \partial_b(K\partial_b \bar{u}) = 0$

$$\underbrace{\cancel{\langle \partial_a(K\partial_a u_2)\rangle_a}}_{FA}^{\,0} + \langle \partial_b(K\partial_b \bar{u})\rangle_a = 0$$

$$\partial_b \langle K\rangle_a \partial_b \bar{u} = 0 \quad \Rightarrow \quad \bar{u}\,(x,y,z,c,t)$$

$\mathcal{O}\left(\frac{1}{\varepsilon^3}\right)$: $\partial_a(K\partial_a u_3) + \partial_b(K\partial_b u_1) + \partial_a(K\partial_x \bar{u}) + \cancel{\partial_x(K\partial_a \bar{u})}^{\,0} = 0$

$$\langle \partial_a(K\partial_a u_3 + K\partial_x \bar{u})\rangle_a + \cancel{\langle \partial_b(K\partial_b u_1)\rangle_a}^{\,0} = 0$$

periodic $\quad$ indep of $a$

$$K\partial_a u_3 + K\partial_x \bar{u} = A \quad \text{indep } a$$

$$\partial_a u_3 + \partial_x \bar{u} = \frac{A}{K}$$

$$\cancel{\langle \partial_a u_3 \rangle_a}^{\to 0} + \langle \partial_x \bar{u} \rangle_a = \left\langle \frac{A}{K} \right\rangle_a$$

$$\partial_x \bar{u} = A \left( \langle K \rangle_a^h \right)^{-1}$$

$$\Rightarrow \boxed{K [ \partial_a u_3 + \partial_x \bar{u} ] = \langle K \rangle_a^h \partial_x \bar{u}}$$ important

$$\Rightarrow u_1(x, y, z, c, t)$$

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$$O(\varepsilon^{-2}): \cancel{\partial_a(K\partial_a u_4)} + \cancel{\partial_b(K\partial_b u_2)} + \partial_a(K\partial_x u_1) + \cancel{\partial_x(K\partial_a u_1)}^{\to 0} + \cancel{\partial_y(K\partial_b \bar{u})}^{\to 0}$$
$$+ \partial_b(K\partial_y \bar{u}) + \partial_c(K\partial_c \bar{u}) = 0$$

$$\underbrace{\cancel{\langle \partial_a (K\partial_a u_4 + K\partial_x u_1) \rangle_a}^{\to 0}}_{FA} + \underbrace{\langle \partial_b (K\partial_b u_2 + K\partial_y \bar{u}) \rangle_a} + \cancel{\langle \partial_c (K\partial_c \bar{u}) \rangle_a}^{\to 0} = 0$$

periodic, indep a

$$\langle \cdot \rangle_b \Rightarrow \langle K \rangle_a [ \partial_b u_2 + \partial_y \bar{u} ] = \langle \langle K \rangle_a \rangle_b^h \partial_y \bar{u} \quad \text{and} \quad \bar{u}(x,y,z,t)$$

$$O(\varepsilon^{-1}): \partial_a(K\partial_a u_5) + \partial_b(K\partial_b u_3) + \partial_a(K\partial_x u_2) + \cancel{\partial_x(K\partial_a u_2)}^{\to 0} + \cancel{\partial_y(K\partial_b u_1)}^{\to 0} + \partial_b(K\partial_y u_1)$$
$$+ \partial_c(K\partial_c u_1) + \cancel{\partial_z(K\partial_c \bar{u})}^{\to 0} + \partial_c(K\partial_z \bar{u}) = 0$$

$$\cancel{\langle \partial_a (K\partial_a u_5 + K\partial_x u_2) \rangle_a}^{\to 0} + \langle \partial_b (K\partial_b u_3 + K\partial_y u_1) \rangle_a + \langle \partial_c (K\partial_c u_1 + K\partial_z \bar{u}) \rangle_a = 0$$

$$\langle \cdot \rangle_b \Rightarrow \cancel{\langle \partial_b (\langle K \rangle_a \partial_b u_3 + \langle K \rangle_a \partial_y u_1 \rangle_b}^{\to 0} + \langle \partial_c (\langle K \rangle_a \partial_c u_1 + \langle K \rangle_a \partial_z \bar{u} \rangle_b = 0$$

$$\langle\langle K \rangle_a \rangle_b (\partial_c u_1 + \partial_z \bar{u}) = \langle\langle\langle K \rangle_a \rangle_b \rangle_c^h \partial_z \bar{u} \quad \text{and}$$

$$\boxed{\begin{array}{l} u_1(x,y,z,c,t) \\ u_2(x,y,z,b,c, \end{array}}$$

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$$O(1): \partial_a(K\partial_a u_6) + \partial_b(K\partial_b u_4) + \partial_a(K\partial_x u_2) + \partial_x(K\partial_a u_3) + \partial_y(K\partial_b u_2) + \partial_b(K\partial_y u_2)$$
$$+ \partial_c(K\partial_c u_2) + \partial_z(K\partial_c u_1) + \partial_c(K\partial_z u_1) + \partial_x(K\partial_x \bar{u}) + \partial_y(K\partial_y \bar{u}) + \partial_z(K\partial_z \bar{u}) = \bar{u}_t$$

$$\langle \cdot \rangle_a \Rightarrow \langle \cdot \rangle_b \Rightarrow \langle \cdot \rangle_c$$

$$\Rightarrow \bar{u}_t = \underbrace{\langle\langle\langle K \rangle_a^h \rangle_b \rangle_c}_{\alpha} \bar{u}_{xx} + \underbrace{\langle\langle\langle K \rangle_a \rangle_b^h \rangle_c}_{\beta} \bar{u}_{yy} + \underbrace{\langle\langle\langle K \rangle_a \rangle_b \rangle_c^h}_{\gamma} \bar{u}_{zz}$$

$$\bar{u} = \mathcal{F}^{-1} \left\{ A e^{-(\alpha \varkappa^2 + \beta \eta^2 + c\mu^2) t} \right\}$$

where $\mathcal{F}(\bar{u}_{xx}) = \varkappa^2$, $\mathcal{F}(\bar{u}_{yy}) = \eta^2$, $\mathcal{F}(\bar{u}_{zz}) = \mu^2$

Aug 2016

# Problem 5

$$f(x,t) = \frac{2xt\left((1-t^2)^2 - x^2\right)}{(1-t^2)^{3/2}\left(t^2(1-t^2)^2 + x^2\right)} = g(x,t)\cdot\frac{(1-t^2)^2 - x^2}{(1-t^2)^{3/2}}$$

$$g(x,t) = \frac{2xt}{t^2(1-t^2)^2 + x^2}$$

a) $$\lim_{t\to 0^+} \frac{f-g}{g} = \lim_{t\to 0^+} \frac{(1-t^2)^2 - x^2}{(1-t^2)^{3/2}} - 1 = 1 - \underbrace{\left(\lim_{t\to 0^+} \frac{x^2}{(1-t^2)^{3/2}}\right)}_{\text{want } = 0} - 1$$

Sps $x \sim t^\delta$

$$\lim_{t\to 0^+} \frac{t^\delta}{(1-t^2)^{3/2}} = 0 \quad \text{if} \quad \delta > 3$$

Sps $x \sim -t^\beta$

$$\lim_{t\to 0^+} \frac{-t^\beta}{(1-t^2)^{3/2}} = 0 \quad \text{if} \quad \beta > 3.$$

$\Rightarrow$ $g$ and $f$ are asymptotic if $x \notin [-t^\beta, t^\delta]$

for $\beta = \delta = 3$.

$\therefore$ $\nexists\ \delta, \beta$ pair st $x \in (-t^\beta, t^\delta) \Rightarrow g \sim f$.

# Problem 6 Aug 2016

$$\begin{cases} T_t + \gamma(2+\cos t)\, x\, T_x = k T_{xx} \\ T(x,0) = T_0(\alpha x) \end{cases} \qquad (1)$$

a) $\alpha x = z$ where $z$ is non-dim

$$\boxed{[\alpha] = \frac{1}{L}}$$

Let $\tau = \frac{t}{T}$

$$\Rightarrow \frac{dT}{dt} = \frac{1}{T}\frac{dT}{d\tau}$$

$$\frac{dT}{dx} = \frac{1}{L}\frac{dT}{dz}$$

So (1) becomes

$$\frac{1}{T} T_\tau + \gamma(2+\cos\tau)\, zL\left(\frac{1}{L} T_z\right) = k\left(\frac{1}{L^2}\right) T_{zz}$$

$$T_\tau + \gamma T(2+\cos\tau) T_z = \frac{kT}{L^2} T_{zz}$$

$\bar{\gamma} = \gamma T$

$\bar{k} = \frac{kT}{L^2}$ both non-dim so $\boxed{\begin{aligned}[\gamma] &= \frac{1}{T} \\ [k] &= \frac{L^2}{T}\end{aligned}}$

b) $$\boxed{T_\tau + \bar{\gamma}(2+\cos\tau)\, z\, T_z = \bar{k} T_{zz}} \qquad (2)$$

is a non-dimensional equation

c) For ease of notation, let's use the origonal notation but note that it is non-dimensional.

$$\widehat{xT_x} = \int_{-\infty}^{\infty} e^{-ikx} \, x T_x \, dx = i \frac{\partial}{\partial k} \int_{-\infty}^{\infty} e^{-ikx} \, T_x \, dx$$

$$= i \frac{\partial}{\partial k} \left( \cancel{e^{-ikx} T \Big|_{-\infty}^{\infty}}^{\to 0} + ik \int_{-\infty}^{\infty} e^{-ikx} \, T \, dx \right) = - \frac{\partial}{\partial k} \left( k \hat{T} \right)$$

$$\widehat{T_t} = \hat{T}_t$$

$$\widehat{T_{xx}} = (-ik)^2 \hat{T} = -k^2 \hat{T}$$

So $\mathcal{F}((1))$ gives:

$$\hat{T}_t - \gamma(2+\cos t) \frac{\partial}{\partial k}(k\hat{T}) = -\kappa k^2 \hat{T} \qquad (3)$$

Let $u = \hat{T}$. Then (3) becomes

$$u_t - \gamma(2+\cos t)(u + k u_k) = -\kappa k^2 u$$

Method of Characteristics

not the T from the eqn

$$\begin{cases} z(s) = u(K(s), T(s)) \\ K(0) = k \\ T(0) = t \end{cases}$$

$$z_s = u_K \frac{dK}{ds} + u_T \frac{dT}{ds}$$

$$z_s = \left( \gamma(2 + \cos T(s)) - \kappa K^2(s) \right) z \qquad \text{from (3)}$$

From coeff's of (3) we know

$$\begin{cases} \frac{dK}{ds} = -\gamma K(s)(2 + \cos T(s)) \\ \\ \frac{dT}{ds} = 1 \end{cases} \Rightarrow \begin{cases} K(s) = k e^{-\gamma(2+\cos(T(s)))} \\ T(s) = s + t \end{cases}$$

$$\frac{dz}{z} = \gamma(2+\cos(s+t)) - kk e^{-2\gamma(2+\cos(s+t))}\, ds$$

$$z(s) = z(0)\, e^{2\gamma + \gamma \underbrace{\int_0^s \cos(v+t)\,dv}_{\sin(s+t)-\sin t} - kk\int_0^s e^{-2\gamma(2+\cos(v+t))}\,dv}$$

$$z(0) = z(s)\, e^{-2\gamma - \gamma(\sin(s+t)-\sin t) + kk e^{-4\gamma}\int_0^s e^{-2\gamma\cos(v+t)}\,dv} \quad (4)$$

Note: $u(k,t) = z(0)$

Let $s=-t$

$$z(-t) = u(ke^{-3\gamma}, 0) = u_0(ke^{-3\gamma}) = \hat{T}_0$$

Using $\mathcal{F}^{-1}$ w/ (4)

$$\boxed{T(x,t) = \int_{\mathbb{R}} e^{ikx}\, \hat{T}_0\, e^{-2\gamma + \gamma\sin t + kk e^{-4\gamma}\int_0^{-t} e^{-2\gamma\cos(v+t)}\,dv}\, dk}$$

d) Take the deriv of (1) wrt x

$$T_{tx} + \gamma\cos t\,(T_x + xT_{xx}) = k\, T_{xxx}$$

Let $u = T_x$

$$u_t + \gamma\cos t\,(u + xu_x) = k\, u_{xx}$$

Proceed as in part b to solve for u.

# Problem 7



a) Let $\Omega \in \mathbb{C}$, $x_0 \in \Omega$, and $f_n$ be a seq of functions

<u>Ptws:</u> We say $f_n \to f$ conv ptws if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ st $\forall n > N$ we have $|f_n(x_0) - f(x_0)| < \varepsilon$.

Only concerned w/ the fixed $x_0$ and $N$ changes.

<u>Asym:</u> We say $f_N \to f$ conv asym'ly if, for fixed $N$, $\forall \varepsilon > 0$ $\exists \delta > 0$ st $|x - x_0| < \delta \Rightarrow \left|\frac{f_N(x) - f(x)}{f(x)}\right| < \varepsilon$.

Only concerned w/ nbhd of $x_0$, not $x_0$ itself, and $N$ is fixed.

<u>Ex:</u>

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ratio Test: $\left|\frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n}\right| = \left|\frac{x}{n+1}\right| \xrightarrow[x \to \infty]{n \text{ fixed}} \infty$ $\therefore$ not ptws conv

$\xrightarrow[x \text{ fixed}]{n \to \infty} 0$ $\therefore$ asym conv

$$\sum_{n=0}^{\infty} \frac{n!}{x^n}$$

Ratio Test: $\left|\frac{(n+1)!}{x^{n+1}} \cdot \frac{x^n}{n!}\right| = \left|\frac{n+1}{x}\right| \xrightarrow[x \to \infty]{n \text{ fixed}} 0$ $\therefore$ ptws conv

$\xrightarrow[x \text{ fixed}]{n \to \infty} \infty$ $\therefore$ not asym conv

b) Let $\{f_n\}$ be a seq of asym functions. Fix $N \in \mathbb{N}$.

We say $f_n \to f$ unif asymp if $\forall \varepsilon > 0 \ \exists \delta$ st

$$|x - x_0| < \delta \Rightarrow \left| \Sigma f_n(x) - f(x_0) \right| < \varepsilon .$$