METHODS OF APPLIED MATHEMATICS COMPREHENSIVE EXAMINATION JANUARY 2013

Work on as many of the following problems as possible. Turn in *all* your work.

(1) Find a Laplace transform solution of the following initial-boundary value problem

$$u_{tt} - u_{xx} - u = 0\,, \qquad u(x,0) = u_t(x,0) = 0\,, \qquad u(0,t) = t^2$$

for the unknown function $u(x,t)$ defined on the half-line $x \geq 0$, with $u(x,\cdot) \to 0$ as $x \to \infty$. Discuss the asymptotic values of the solution for large values of x and t, for both cases $x > t$ and $x < t$.

(2) Apply iterated homogenization (average fastest scale first and repeat), on the following initial value problem to derive a leading order effective equation as $\epsilon \to 0^+$, assuming the functions $K(x), M(x)$ are periodic sharing the same period, and positive:

$$\begin{aligned} u_t &= \left[\left(K(x/\epsilon) + M(x/\epsilon^2)\right) u_x\right]_x \\ u(x,t=0) &= u_0(x) \end{aligned}$$

Make sure to explicitly express the effective diffusion coefficient in terms of the functions K and M.

(3) Find the first term in the asymptotic expansion of

$$\int_{c-i\infty}^{c+i\infty} \frac{1}{s^3} \exp\left(t(s - m\sqrt{s^2-1})\right) ds, \qquad c > 0\,,$$

as $t \to \infty$, with fixed $m > 0$. Clearly state the assumptions on the parameter c and the choice of branch cuts for the contour integral to be well defined. Discuss what happens for each of the two cases: (i) $0 < m < 1$, and (ii) $m > 1$.

(4) Consider a driven heat equation on an interval with $x \in [-1,1]$ for $t \geq 0$:

$$\begin{aligned} c_t - c_{xx} &= e^{-at}\cos(n\pi x) \\ c_x|_{x=-1,1} &= 0 \\ c(0,t) &= \cos(2\pi x) \end{aligned}$$

for a positive real number a, and an integer $n > 0$. Find the long time asymptotics of the solution and find all critical values of the parameter a which modify the long time asymptotics. What happens if n is not an integer? Discuss.

(5) Use either multiscale methods, or method of averaging to discuss the behavior of the following variable frequency oscillator for small values of a parameter $\epsilon > 0$:

$$c''' + \omega^2(\epsilon t)c' = 0$$

Are periodic solutions possible? Discuss behavior for $\omega(z) = a + b\cos(z)$, and $a > b > 0$. What happens if $a < b$? Discuss. What about the limit $a \to b$ from above? Discuss.

(6) Find a two term asymptotic expansion for all the (complex) roots of the following as $\epsilon \to 0$,

$$(x-1)^2 + \epsilon x^{5/2} = 0\,.$$

(7) Compute the leading order term in an asymptotic expansion of the solution $y(x)$ of the following fourth order differential equation

$$y'''' = xy + 2y$$

as $x \to \pm\infty$. Classify the different forms of the solutions with respect to their asymptotic behavior, and discuss their linear (in)dependency.

(8) (a) Explain the difference between pointwise convergence and asymptotic convergence. Illustrate with the particular example of power series.

(b) Are asymptotic series unique? Explain

(c) Does an asympotic relation carry over to products? Discuss in the context of the following example: does

$$f(x) \sim g(x) \quad \text{and} \quad h(x) \sim k(x)\,, \qquad \text{as} \quad x \to \infty\,,$$

imply

$$f(x)h(x) \sim g(x)k(x)\ ?$$

Make sure to clarify your assumptions on all functions f, g, h and k for a definitive, but non trivial, statement.

Problem 1

$$\begin{cases} u_{tt} - u_{xx} - u = 0 \\ u(x,0) = u_t(x,0) = 0 \\ u(0,t) = t^2 \end{cases}$$

$\mathcal{L}(u) = \hat{u}$

$$\mathcal{L}(u_{tt}) = \int_0^\infty e^{-st} \frac{d^2}{dt^2} u(x,t)\, dt = \cancel{e^{-st} u_t \Big|_0^\infty} \xrightarrow{} 0 + s\int_0^\infty e^{-st} u_t\, dt$$

$$= \cancel{se^{-st} u \Big|_0^\infty} \to 0 + s^2 \int_0^\infty e^{-st} u\, dt = s^2 \hat{u}$$

$\mathcal{L}(u_{xx}) = \hat{u}_{xx}$

$$\Rightarrow \quad s^2 \hat{u} - \hat{u}_{xx} - \hat{u} = 0$$

$$\hat{u}_{xx} = (s^2 - 1)\hat{u}$$

$$\hat{u} = A(s) e^{-x\sqrt{s^2-1}} + B(s) e^{x\sqrt{s^2-1}}$$

But since $u(x,t) \longrightarrow 0$ as $x \longrightarrow \infty$, we must set $B(s) = 0$ to prevent our soln from tending to ∞..

$$\hat{u}(x,s) = A(s) e^{-x\sqrt{s^2-1}}$$

$$A(s) = \hat{u}(0,s) = \int_0^\infty e^{-st} t^2\, dt = -\cancel{\frac{e^{-st} t^2}{s}\Big|_0^\infty} \to 0 + \frac{1}{s}\int_0^\infty e^{-st} t\, dt$$

$$= -\cancel{\frac{e^{-st} t}{s^2}\Big|_0^\infty} \to 0 + \frac{1}{s^2}\int_0^\infty e^{-st}\, dt = -\frac{1}{s^3} e^{-st}\Big|_0^\infty = \frac{1}{s^3}$$

$$\boxed{\hat{u}(x,s) = \frac{1}{s^3} e^{-x\sqrt{s^2-1}}}$$

$$u(x,t) = \frac{1}{2\pi i} \int_0^\infty e^{st} \left(\frac{1}{s^3} e^{-x\sqrt{s^2-1}} \right) ds$$

$$\Rightarrow \quad u(x,t) = \frac{1}{2\pi i} \int_0^\infty \frac{1}{s^3} e^{st - x\sqrt{s^2-1}} \, ds$$

Problem 1

Jan 2013

$$\begin{cases} u_{tt} - u_{xx} - u = 0 \\ u(x,0) = u_t(x,0) = 0 \\ u(0,t) = t^2 \end{cases}$$

$\mathcal{L}(u) = \hat{u}$

$$\mathcal{L}(u_{tt}) = \int_0^\infty e^{-st} \frac{d^2}{dt^2} u(x,t)\, dt = \cancelto{0}{e^{-st} u_t(x,t)\Big|_0^\infty} + s\int_0^\infty e^{-st} \frac{d}{dt} u \, dt$$

$$= s\left(\cancelto{0}{e^{-st} u(x,t)\Big|_0^\infty} + s \int_0^\infty e^{-st} u(x,t)\, dt \right)$$

$$= s^2 \hat{u}$$

$$\mathcal{L}(u_{xx}) = \hat{u}_{xx}$$

$$s^2 \hat{u} - \hat{u}_{xx} - \hat{u} = 0$$

$$\hat{u}_{xx} + (1-s^2)\hat{u} = 0$$

$$r^2 + (1-s^2) = 0$$

$$r = \pm\sqrt{s^2-1}$$

$$\hat{u}(x,s) = A(s) e^{-x\sqrt{s^2-1}} + B(s) e^{+x\sqrt{s^2-1}}$$

However to have $u(x,t) \to 0$ as $x \to \infty$, we must set $B(s) = 0$ else our soln will tend to ∞.

$$\hat{u}(x,s) = A(s) e^{-x\sqrt{s^2-1}}$$

$$A(s) = \hat{u}(0,s) = \int_0^\infty e^{-st} t^2 \, ds = \cancelto{0}{t^2(-\tfrac{1}{s})e^{-st}\Big|_0^\infty} + \frac{1}{s}\int_0^\infty e^{-st} t$$

$$= \frac{1}{s}\left(t(-\tfrac{1}{s})e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt \right) = -\frac{1}{s^3} e^{-st}\Big|_0^\infty = \frac{1}{s^3}$$

$$\boxed{\hat{u}(x,s) = \frac{1}{s^3} e^{-x\sqrt{s^2-1}}}$$

$$\Rightarrow \quad u(x,s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st - x\sqrt{s^2-1}}}{s^3} \, ds$$

Any analysis done for large t is analygous to problem (3) and will not be done here. Thus we need only consider large x.

Problem 2

$$u_t = \partial_x\Big(\underbrace{\left(K(\tfrac{x}{\varepsilon}) + M(\tfrac{x}{\varepsilon^2})\right)}_{L(x,\varepsilon)} u_x\Big)$$

$$u\big|_{t=0} = u_0(x)$$

Let $\omega = \frac{x}{\varepsilon^2}$, $z = \frac{x}{\varepsilon}$ $\Rightarrow$ $\partial_x \mapsto \partial_x + \frac{1}{\varepsilon}\partial_z + \frac{1}{\varepsilon^2}\partial_\omega$

Ansatz: $u(x,\omega,z,t) = \bar{u}(x,\omega,z,t) + \varepsilon u_1(x,\omega,z,t) + \dots$

$$u_t = \left(\partial_x + \tfrac{1}{\varepsilon}\partial_z + \tfrac{1}{\varepsilon^2}\partial_\omega\right)\left[L(\omega,z)\left(\partial_x + \tfrac{1}{\varepsilon}\partial_z + \tfrac{1}{\varepsilon^2}\partial_\omega\right)u\right]$$

$O(\frac{1}{\varepsilon^4})$: $\partial_\omega(L\partial_\omega \bar{u}) = 0 \Rightarrow \bar{u}(x,z,t)$

$O(\frac{1}{\varepsilon^3})$: $\partial_\omega(L\partial_\omega u_1) + \cancelto{0}{\partial_z(L\partial_\omega \bar{u})} + \partial_\omega(L\partial_z\bar{u}) = 0$

$$\langle \partial_\omega(L\partial_\omega u_1 + L\partial_z\bar{u})\rangle_\omega = 0$$

$$L\partial_\omega u_1 + L\partial_z\bar{u} = A(x,z,t)$$

$$\langle \partial_\omega u_1\rangle_\omega + \langle \partial_z \bar{u}\rangle_\omega = \left\langle \frac{A}{L}\right\rangle_\omega$$

$$A = \langle L\rangle_\omega^h \partial_z\bar{u}$$

$$L(\partial_\omega u_1 + \partial_z\bar{u}) = \langle L\rangle_\omega^h \partial_z\bar{u}$$

$O(\frac{1}{\varepsilon^2})$: $\partial_\omega(L\partial_\omega u_2) + \partial_z(L\partial_\omega u_1) + \partial_\omega(L\partial_z u_1) + \partial_z(L\partial_z\bar{u}) = 0$

$$\cancelto{0}{\langle\partial_\omega(L\partial_\omega u_2 + L\partial_z u_1)\rangle_\omega} + \langle\partial_z(L\partial_\omega u_1)\rangle_\omega + \langle\partial_z(L\partial_z\bar{u})\rangle = 0$$

$\hookrightarrow L\partial_\omega u_2 + L\partial_z u_1 = \langle L\rangle_\omega^h \partial_z u_1$

$$\Rightarrow u_1(x,z,t)$$

$$\Rightarrow \bar{u}(x,t)$$

$$O(1/\varepsilon):\ \partial_w(L\partial_w u_3) + \partial_z(L\partial_w u_2) + \partial_w(L\partial_z u_2) + \partial_z(L\partial_z u_1) + \partial_z(L\partial_x \bar{u}) + \underbrace{\cancel{\partial_x(L\partial_z \bar{u})}}_{0} = 0$$

$$\underbrace{\cancel{\langle \partial_w(L\partial_w u_3 + L\partial_z u_2)\rangle_w}}_{0} + \langle \partial_z(L\partial_w u_2)\rangle_w + \langle L\partial_z u_1 + L\partial_x \bar{u})\rangle_w = 0$$

$$\underbrace{\cancel{\langle \partial_z \langle L\partial_w u_2\rangle_w\rangle_z}}_{0} + \langle \langle L\rangle_w(\partial_z u_1 + \partial_x \bar{u})\rangle_z = 0$$

$$\Rightarrow \langle L\rangle_w(\partial_z u_1 + \partial_x \bar{u}) = \langle\langle L\rangle_w\rangle_z^h\, \partial_x \bar{u}$$

$$O(1):\ \partial_w(L\partial_w u_4) + \partial_z(L\partial_w u_3) + \partial_w(L\partial_z u_3) + \partial_z(L\partial_z u_2) + \partial_z(L\partial_x u_1) + \partial_x(L\partial_z u_1) + \partial_x(L\partial_x \bar{u}) = \bar{u}_t$$

Avg wrt w

$$\Rightarrow \partial_z[\langle L\rangle_w(\partial_z u_2 + \partial_x u_1)] + \partial_x[\underbrace{\langle L\rangle_w(\partial_z u_1 + \partial_x \bar{u})}_{\langle\langle L\rangle_w\rangle_z^h \bar{u}_x}] = \bar{u}_t$$

Avg wrt z

$$\langle \partial_z[\langle L\rangle_w(\partial_z u_2 + \partial_x u_1)]\rangle_z + \langle \partial_x[\langle\langle L\rangle_w\rangle_z^h\, \bar{u}_x]\rangle_z = \langle \bar{u}_t\rangle_z$$

$$\therefore \boxed{\bar{u}_t = \langle\langle L\rangle_w\rangle_z^h\, \bar{u}_{xx}}$$

Problem 5

Jan 2013

$$c''' + \omega^2(\varepsilon t)\, c' = 0 \qquad (1)$$

Let $y = c'$

$$\Rightarrow \quad y'' + \omega^2(\varepsilon t)\, y = 0 \qquad (2)$$

$$\tau = \varepsilon t \quad \Rightarrow \quad \frac{d\tau}{dt} = \varepsilon$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{dy}{d\tau}\frac{d\tau}{dt} = \varepsilon\, y_\tau$$

$$\varepsilon^2 y_{\tau\tau} + \omega^2(\tau)\, y = 0$$

$$\varepsilon^2 \ddot{y} = -\omega^2(\tau)\, y \qquad \text{(This is a WKB problem)}$$

$$s_0 = w \int^{\tau} (-\omega^2(x))^{1/2}\, dx \qquad w^2 = 1$$

$$s_1 = -\tfrac{1}{4} \log(-\omega^2(\tau))$$

$$\Rightarrow \quad y \sim e^{\frac{s_0}{\varepsilon} + s_1}$$

Problem 6

Jan 2013

$$(x-1)^2 + \varepsilon x^{5/2} = 0 \qquad (1)$$

Consider the unpreturbed problem:

$(x-1)^2 = 0$

$x = 1, 1$

$x_0 = 1 + a_1 \varepsilon^{1/2} + a_2 \varepsilon + \dots$

Sub into eqn 1:

$(1 + a_1\varepsilon^{1/2} + a_2\varepsilon + \dots - 1)^2 + \varepsilon(1 + a_1\varepsilon^{1/2} \dots)^{5/2} = 0$

$(a_1\varepsilon^{1/2})^2 + \varepsilon(1 + a_1\varepsilon^{1/2})^{5/2} = 0$

$(a_1\varepsilon^{1/2})^2 = -\varepsilon(1 + a_1\varepsilon^{1/2})^{5/2} \qquad *$

$a_1^4\varepsilon^2 = \varepsilon^2(1 + a_1\varepsilon^{1/2})^5$

$a_1^4 - (1 + a_1\varepsilon^{1/2})^5 = 0$

$a_1^4 - 1 + 5a_1\varepsilon^{1/2} + 10a_1^2\varepsilon + 10a_1^3\varepsilon^{3/2} + 5a_1^4\varepsilon^2 + a_1^5\varepsilon^{5/2} = 0$

$O(1)$: $a_1^4 - 1 = 0$

$a_1 = \pm 1, \pm i$ But we should only have 2 roots

The extra roots come from picking a Branch cut.

Consider $(x-1) = \rho_1 e^{i\theta_1}$ and $x = \rho_0 e^{i\theta_0}$

$\Rightarrow \rho_1^2 e^{i2\theta_1} = \varepsilon\rho_0 e^{i\theta_0}$

As $\varepsilon \to 0 \Rightarrow \rho_1^2 e^{i2\theta_1} = 0 \Rightarrow 2\theta_1 \to \pm\pi$ to keep $x-1$ real

From $*$:

$\varepsilon(x-1)^{5/2} = -a_1^2\varepsilon$

$\rho_1^2 e^{i2\theta_1} = -a_1^2$

Pos branch: $a_1^2 < 0 \Rightarrow a_1^2 = -1 \Rightarrow a_1 = \pm i \qquad x_{1,2} = 1 \pm i\varepsilon^{1/2}$

Neg branch: $a_1^2 > 0 \Rightarrow a_1^2 = 1 \Rightarrow a_1 = \pm 1 \qquad x_{1,2} = 1 \pm \varepsilon^{1/2}$

Problem 6

Jan 2013

$$(x-1)^2 + \varepsilon x^{5/2} = 0 \qquad (1)$$

Let $\varepsilon = 0 \Rightarrow (x-1)^2 = 0$

$x = 1$ degenerate

$\Rightarrow x_{1,2} \sim 1 + a\varepsilon^{\alpha}$

Plug into (1):

$$(1 + a\varepsilon^{\alpha} - 1)^2 + (1 + a\varepsilon^{\alpha})^{5/2} \cdot \varepsilon = 0$$

$$a^2 \varepsilon^{2\alpha} + \varepsilon(1 + a\varepsilon^{\alpha})^{5/2} = 0$$

$$\varepsilon(1 + a\varepsilon^{\alpha})^{5/2} = -a^2 \varepsilon^{2\alpha}$$

$$\varepsilon^2 (1 + a\varepsilon^{\alpha})^5 = a^4 \varepsilon^{4\alpha}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$\varepsilon^2(1 + 5a\varepsilon^{\alpha} + 10a^2\varepsilon^{2\alpha} + 10a^3\varepsilon^{3\alpha} + 5a^4\varepsilon^{4\alpha} + a^5\varepsilon^{5\alpha}) = a^4 \varepsilon^{4\alpha}$$

$5\alpha + 2 = 4\alpha$

$\alpha = -2 \Rightarrow \varepsilon \gg 1$!!

$2 = 4\alpha$

$\alpha = \frac{1}{2} \Rightarrow$ consistent

$O(\varepsilon^2)$: $1 = a^4$

$a = \pm 1, \pm i$ We have 4 answers when we should only have 2 b/c we have 2 answers for each branch of $\sqrt{\ }$.

$x \sim \rho_1 e^{i\theta_1}$

$x - 1 \sim \rho_2 e^{i\theta_2} \Rightarrow \rho_2^2 e^{i2\theta_2} = -\varepsilon \rho_1 e^{i\theta_1}$

$$\boxed{\begin{array}{ll} x_{1,2} \sim 1 \pm i\varepsilon^{1/2} & \text{pos branch} \\ x_{1,2} \sim 1 \pm \varepsilon^{1/2} & \text{neg branch} \end{array}}$$