METHODS OF APPLIED MATHEMATICS COMPREHENSIVE EXAMINATION JANUARY 2015

Work on as many of the following problems as possible. Turn in *all* your work.

(1) Consider a projectile fired vertically from the surface of some large body. Assume there is an inverse *quartic*, attractive gravitational force field ($F = -\frac{GM_e m}{R^4}$, where R distance between the projectile and the large body's center, M_e, m are the respective body and projectile mass, and G is a constant.) Assume the projectile experiences a drag force proportional to the square of its velocity.

(a) Derive an equation for the projectile's dynamics assuming that the large body is fixed, decouple the dynamics. Take care with the sign of the drag force.

(b) Measure the projectile height from the body surface by introducing R_e as the radius of the body. Then, identify standard kinematics by letting $g = \frac{GM_e}{R_e^4} \approx 9.8 m/s^2$. Identify characteristic length and time scales in terms of g and the initial velocity, v_0. Non-dimensionalize the problem, and identify the non-dimensional parameters and simplified non-linear ODE governing the evolution.

(c) Is there an escape velocity for this problem in the absence of drag? If so, find it.

(d) Assuming small enough initial projectile speeds to guarantee return, and that the drag is weak, find via perturbation expansion, the first correction for the time to return to the body surface beyond leading order kinematic predictions.

(2) Solve the heat equation $u_t = \kappa u_{xx}$ using Laplace transform, assuming vanishing initial data, $u(x, 0) = 0$, and boundary data $u(0, t) = t^2$ for

(a) for the finite interval $x \in [0, L]$, with $u(L, t) = 0$.

(b) for the semi-infinite interval $x \in [0, \infty)$, with $u(x, t) \to 0$, as $x \to +\infty$.

Discuss the differences in the branch structure of the solution's Laplace transform. Compute the long time asymptotics for each case. Sketch the solutions at long time.

(3) Consider the rapidly varying diffusivity:

$$K(x, y; \epsilon) = A + F(x/\epsilon^2) + G(y/\epsilon)$$

where A is chosen to guarantee K is positive, and ϵ is a small constant. By applying homogenization using two fast scales, average the following diffusion equation to derive a leading order effective equation governing the evolution as $\epsilon \to 0$ over the (x, y)-plane, assuming the functions $F(x)$ and $G(y)$ are mean zero, periodic, and share the same period:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K(x,y;\epsilon)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K(x,y;\epsilon)\frac{\partial u}{\partial y}\right)$$
$$u(x,y,0) = u_0(x,y)\,.$$

Compare this result with the result obtained using more formal iterated homogenization in which the fastest scales are first averaged, while holding the slower scales constant, and then repeating.

(4) Find three term asymptotic expansions as $\epsilon \to 0$ for the unperturbed root at the origin of the equation

$$\epsilon z^3 + z^2 = \epsilon \cos z\,.$$

Discuss the possibility of other roots of this equation, and how they behave as $\epsilon \to 0$.

(5) Consider the partial differential equation

$$\phi_t + \phi_{xxx} - \epsilon\phi_{xx} = 0, \qquad 0 < \epsilon \ll 1\,.$$

(a) Find the dispersion relation $\omega = W(k)$ for simple sinusoidal solutions

$$\phi(x,t) = e^{i(kx-\omega t)}\,.$$

(b) Find the asymptotic behavior for $t \to \infty$ along rays $x/t = m$, m fixed, of the general solution of the initial value problem for this equation

$$\phi(x,t) = \int_{-\infty}^{+\infty} F(k)e^{i(kx-W(k)t)}dk$$

with $F(k)$) a smooth function of k, noting the different behavior (oscillary vs. exponential decay) in x, fixed (large) t. Comment on the limit $\epsilon \to 0$.

(6) Consider the eigenvalue problem on the real line $x \in \mathbb{R}$

$$\epsilon y'' - U(x)y = \lambda y.$$

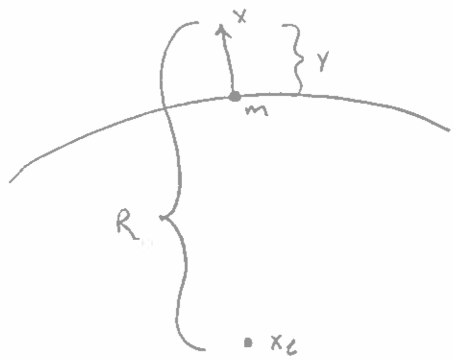
with the (square well) potential $U(x) = 1$ for $|x| > 1$ and $U(x) = -1$ for $|x| < 1$

(a) Solve for eigenvalues and eigenfunctions exactly.
(b) Study their asymptotic limit as $\epsilon \to 0$.
(c) Compare with the results obtained by a WKBJ approach in this limit.

(7) (a) Explain the difference between pointwise convergence and asymptotic convergence. Illustrate with the particular example of power series.
(b) Consider the map that associates to a function $f(z)$ its asymptotic expansion as $z \to z_0$, where z_0 is a point of the complex plane, including infinity. Is this mapping one-to-one (injective), onto (surjective), one-to-one and onto (bijective), or none of these properties applies? Explain.
(c) Is the asymptotic relation distributive? Discuss.

Problem 1 Jan 2015

a)



$$m\ddot{x} = -\frac{GmM}{|x - x_e|^2} - \beta(\dot{x})^3 \qquad (1)$$

$$M\ddot{x}_e = \frac{GmM}{|x - x_e|^2}$$

The vel of an obj is inv proportional to its mass. Since $m_e \gg m$

$\Rightarrow$ $\dot{x}_e \ll \dot{x}$ thus we can say $\dot{x}_e$ is relatively const thus $\ddot{x}_e = 0$.

$$\Rightarrow \begin{cases} \ddot{x} = -\dfrac{GM}{|x - x_e|^2} - \dfrac{\beta}{m}(\dot{x})^3 \\ x(t)\big|_{t=0} = R_e = \text{radius of Earth} \\ \dot{x}\big|_{t=0} = v_0 > 0 \end{cases}$$

b) Change of variables:

$$\begin{cases} y = x - R_e \quad \Rightarrow \quad x = y + R_e \\ x_e = 0 \end{cases}$$

$$\ddot{y} = -\frac{GM}{|y + R_e|^2} - \frac{\beta}{m}(\dot{y})^3$$

$$\boxed{\ddot{y} = -\underbrace{\frac{GM}{R_e^2}}_{g = \text{gravity}} \cdot \frac{1}{|1 + y/R_e|^2} - \frac{\beta}{m}(\dot{y})^3} \qquad (2)$$

where $y(t)\big|_{t=0} = x\big|_{t=0} - R_e = 0$ and

$$\dot{y}\big|_{t=0} = \dot{x}\big|_{t=0} = V_0$$

We know $y \ll R_e \Rightarrow \dfrac{y}{R_e} \approx 0$

$$\Rightarrow \quad \ddot{y} = -g$$

$$\dot{y} = -gt + v_0$$

$$y = -\tfrac{1}{2}gt^2 + v_0 t + 0 = -\tfrac{1}{2}gt^2 + v_0 t$$

$$\dot{y} = 0 \quad \Rightarrow \quad -gt_a + v_0 = 0 \quad \Rightarrow \quad t_a = \frac{v_0}{g}$$

$$\ddot{y}(t_a) = -\frac{1}{2}\, g\, \frac{v_0^2}{g^2} + v_0 \frac{v_0}{g} = \frac{v_0^2}{2g}$$

Choose $L = \dfrac{v_0^2}{g}$ and $T = \dfrac{v_0}{g}$ and let $y = zL$ and $t = \tau T$

non-dim

$$\dot{y} = \frac{d}{dt}(zL) = L\frac{dz}{d\tau}\frac{d\tau}{dt} = \frac{L}{T}\dot{z}$$

$$\ddot{y} = \frac{d}{dt}\left(\frac{L}{T}\frac{dz}{d\tau}\right) = \frac{L}{T}\frac{d^2z}{d\tau^2}\frac{d\tau}{dt} = \frac{L}{T^2}\ddot{z}$$

$$\Rightarrow \quad \underbrace{\frac{L}{T^2}}_{g}\ddot{z} = -g\frac{1}{\left(1+\frac{y}{R_e}\right)^2} - \frac{\beta L^3}{mT^3}(\dot{z})^3$$

$$\ddot{z} = \frac{1}{\left(1+\frac{zL}{R_e}\right)^2} - \frac{\beta L^3}{mT^3}(\dot{z})^3$$

Let $\varepsilon = \dfrac{L}{R_e} = \dfrac{v_0^2}{gR_e}$ and $\alpha = \dfrac{\beta L^2 R_e}{mT^3}$

$$\Rightarrow \quad \boxed{\ddot{z} = -\frac{1}{(1+\varepsilon z)^2} + \alpha\varepsilon(\dot{z})^3} \qquad (3)$$

$$v_0 = \dot{y}\big|_{t=0} = \frac{L}{T}\dot{z}\big|_{\tau=0} = v_0\,\dot{z}\big|_{\tau=0} \quad \Rightarrow \quad \dot{z}\big|_{\tau=0} = 1$$

$$z\big|_{t=0} = \frac{1}{L}\,y\big|_{t=0} = 0 \quad \Rightarrow \quad z(0) = 0$$

c) If there is no drag, then $\ddot{z} = -\frac{1}{(1+\varepsilon z)^2}$

$$\dot{z} \cdot \ddot{z} = -\frac{1}{(1+\varepsilon z)^2}\dot{z}$$

$$2\frac{d}{d\tau}\left((\dot{z})^2\right) = \frac{1}{\varepsilon}\frac{d}{d\tau}\left(\frac{1}{(1+\varepsilon z)}\right)$$

Recall $v = \dot{z} \Rightarrow \frac{dv}{d\tau} = \frac{d^2 z}{d\tau}$

$$\Rightarrow \ddot{z} = \dot{v}\dot{z} = \frac{1}{2}(v^2)_z = -\frac{1}{(1+\varepsilon z)^2}$$

Integrate wrt z

$$v^2(z) - v^2(0) = \int_0^z \frac{-2}{(1+\varepsilon w)^2}\,dw$$

$$v^2(z) = 1 + \frac{1}{\varepsilon(1+\varepsilon z)} - \frac{2}{\varepsilon} = \frac{\varepsilon(1+\varepsilon z) + 1 - 2(1+\varepsilon z)}{\varepsilon(1+\varepsilon z)}$$

$$\dot{z} = v(z) = \sqrt{\frac{\varepsilon + \varepsilon^2 z + 1 - 2 - 2\varepsilon z}{\varepsilon(1+\varepsilon z)}}$$

z_{apex} occurs when $v(z) = 0$

$$\Rightarrow \varepsilon + \varepsilon^2 z - 1 - 2\varepsilon z = 0$$

$$z_{apex} = \frac{1-\varepsilon}{\varepsilon(\varepsilon - 2)}$$

ε_{escape} occurs when z_{apex} DNE so $\varepsilon_{escape} = 2$

d) For sufficiently small v_0 we get $\varepsilon \ll 1$. So consider

$$z \sim z_0 + \varepsilon z_1 + \varepsilon^2 z_2 + \dots$$

Note $\dot{z} = v$ so $\varepsilon(\dot{z})^3$ is negligable.

Plugging this info into (3):

$$(\ddot{z}_0 + \varepsilon \ddot{z} + \dots) = -\frac{1}{(1+\varepsilon(z_0 + \varepsilon z_1))^2} = -1 + 2\varepsilon(z_0 + \varepsilon z_1 + \dots)$$

$O(1)$: $\ddot{z}_0 = -1$

$\dot{z}_0 = -\tau + a \qquad \dot{z}_0(0) = a = 1$

$z_0 = -\frac{1}{2}\tau^2 + \tau + b \qquad z_0(0) = b = 0$

$O(\varepsilon)$: $\ddot{z}_1 = 2 z_0 = \tau^2 + 2\tau$

$\dot{z}_1 = \frac{1}{3}\tau^3 + \tau^2 + c \qquad \dot{z}_1(0) = c = 0$

$z_1 = \frac{1}{12}\tau^4 + \frac{1}{3}\tau^3 + d \qquad z_1(0) = d = 0$

$$z \sim -\frac{1}{2}\tau^2 + \tau + \varepsilon\left(\frac{1}{12}\tau^4 + \frac{1}{3}\tau^3\right) \qquad (4)$$

The return time fulfills $z(\tau_R) = 0$

Ansantz: $\tau_R = \tau_0 + \varepsilon\tau_1 + \dots$

Plug into $(4) = 0$:

$$-\frac{1}{2}(\tau_0 + \varepsilon\tau_1)^2 + \tau_0 + \varepsilon\tau_1 + \varepsilon\left(\frac{1}{12}(\tau_0 + \varepsilon\tau_1)^4 + \frac{1}{3}(\tau_0 + \varepsilon\tau_1)^3\right) = 0$$

$O(1)$: $-\frac{1}{2}\tau_0^2 + \tau_0 = 0 \Rightarrow \tau_0\left(1 - \frac{1}{2}\tau_0\right) = 0 \Rightarrow \tau_0 = 0, 2$

start return

$O(\varepsilon)$: $-\tau_1\tau_0 + \tau_1 + \frac{1}{12}\tau_0^4 + \frac{1}{3}\tau_0^3 = 0$

$-2\tau_1 + \tau_1 + \frac{4}{3} + \frac{8}{3} = 0 \Rightarrow \tau_1 = \frac{12}{3} = 4$

$$\boxed{\tau_R = 2 + 4\varepsilon}$$

Version 2 and more correct version

c) If there is no drag then $\ddot{z} = -\frac{1}{(1+\varepsilon z)^2}$

$$\ddot{z}\dot{z} = -\frac{\dot{z}}{(1+\varepsilon z)^2}$$

$$\frac{1}{2}\frac{d}{d\tau}(\dot{z}^2) = \frac{d}{d\tau}\frac{1}{\varepsilon(1+\varepsilon z)}$$

Recall $v = \dot{z}$ so:

$$v^2 = \frac{2}{\varepsilon(1+\varepsilon z)} + c$$

$$v(0)^2 = \frac{2}{\varepsilon(1+\varepsilon z(0))} + c$$

$$1 = \frac{2}{\varepsilon} + c \quad \Rightarrow \quad c = 1 - \frac{2}{\varepsilon}$$

$$v = \sqrt{\frac{2}{\varepsilon(1+\varepsilon z)} + 1 - \frac{2}{\varepsilon}} = \sqrt{\frac{2+\varepsilon+\varepsilon^2 z - 2 - 2\varepsilon z}{\varepsilon(1+\varepsilon z)}}$$

$$= \sqrt{\frac{1+\varepsilon z - 2z}{\varepsilon(1+\varepsilon z)}}$$

z_{apex} occurs when $v(\tau) = 0$

$$\Rightarrow 1 + (\varepsilon - 2)z = 0$$

$$z_{apex} = \frac{1}{2-\varepsilon}$$

ε_{escape} occurs when z_{apex} DNE so

$$\varepsilon_{escape} = 2$$

Problem 2

a) $$\begin{cases} u_t = k\,u_{xx} \\ u(x,0) = 0 \\ u(0,t) = t^2, \quad u(L,t) = 0 \end{cases}$$

$$\mathcal{L}(u_t) = \int_0^\infty e^{-st} \frac{d}{dt} u(x,t)\,dt = \cancel{e^{-st} u(x,t)\Big|_0^\infty}^{\,\to 0} + s\int_0^\infty e^{-st} u(x,t)\,dt$$

$$\widehat{u_t} = s\,\hat{u}$$

$$\mathcal{L}(u_{xx}) = \int_0^\infty e^{-st} \frac{d^2}{dx^2} u(x,t)\,dt = \frac{d^2}{dx^2}\int_0^\infty e^{-st} u(x,t)\,dt = \hat{u}_{xx}$$

$$\Rightarrow \quad s\hat{u} = k\,\hat{u}_{xx} \quad \Rightarrow \quad \hat{u}_{xx} = \frac{s}{k}\hat{u}$$

$$\hat{u} = A(s)\,e^{-\sqrt{s/k}\,x} + B(s)\,e^{\sqrt{s/k}\,x}$$

$\hat{u}$ is analytic everywhere except $s=0$ which is a b.p.

So choose pos. branch of $\sqrt{\ }$.

$$\hat{u}(L,s) = \int_0^\infty e^{-st} u(L,t)\,dt = 0 = A(s)\,e^{-\sqrt{s/k}\,L} + B(s)\,e^{\sqrt{s/k}\,L}$$

$$\hat{u}(0,s) = \int_0^\infty e^{-st} t^2\,dt = \cancel{-\frac{e^{-st} t^2}{s}\Big|_0^\infty}^{\,\to 0} + 2\int_0^\infty \frac{e^{-st}\,t}{s}\,dt$$

$$= \cancel{\frac{-2e^{-st}\,t}{s^2}\Big|_0^\infty}^{\,\to 0} + \frac{2}{s^2}\int_0^\infty e^{-st}\,dt$$

$$= -\frac{2}{s^3} e^{-st}\Big|_0^\infty = \frac{2}{s^3} = A(s) + B(s)$$

$$\begin{cases} A + B = 2/s^3 \\ A e^{-\sqrt{s/k}\,L} + B e^{\sqrt{s/k}\,L} = 0 \end{cases} \Rightarrow A e^{-\sqrt{s/k}\,L} + \left(\tfrac{2}{s^3} - A\right) e^{\sqrt{s/k}\,L} = 0$$

$$A = \frac{2}{s^3\left(e^{\sqrt{s/k}\,L} - e^{-\sqrt{s/k}\,L}\right)}$$

$$\Rightarrow \quad B = \frac{2}{s^3} - \frac{2}{s^3\left(e^{\sqrt{s/k}\,L} - e^{-\sqrt{s/k}\,L}\right)}$$

$$\hat{u}(x,s) = \frac{2}{s^3\left(e^{\sqrt{s/k}\,L} - e^{-\sqrt{s/k}\,L}\right)} e^{-\sqrt{s/k}\,x} + \frac{2\left(e^{\sqrt{s/k}\,L} - e^{-\sqrt{s/k}\,L} - 1\right)}{s^3\left(e^{\sqrt{s/k}\,L} - e^{-\sqrt{s/k}\,L}\right)} e^{\sqrt{s/k}\,x}$$

$$\boxed{u(x,t) = \frac{1}{2\pi i}\int_0^\infty e^{st}\,\hat{u}(x,s)\,ds}$$

b) If $x \in [0,\infty)$, then we must change our soln to be physically relevant so $u(x,t) \to 0$ as $x \to \infty$.

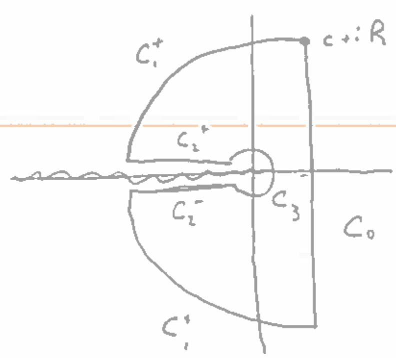
$$\Rightarrow \hat{u}(x,s) = A(s)\,e^{-\sqrt{s/k}\,x}$$

$$\hat{u}(0,s) = \frac{2}{s^3} = A(s) \quad \Rightarrow \quad \hat{u}(x,s) = \frac{2}{s^3} e^{-\sqrt{s/k}\,x}$$

$$\Rightarrow u(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\left(\frac{2}{s^3} e^{-\sqrt{s/k}\,x}\right) ds = \frac{1}{\pi i}\underbrace{\int_{c-i\infty}^{c+i\infty} \frac{e^{st-\sqrt{s/k}\,x}}{s^3}\,ds}_{I_1}$$

There is a triple pole at $s=0$ so first, one should perform integration by parts (probably twice). The $uv\big|_0^\infty$ will probably tend to zero as $t\to\infty$ and we'll eventually find $I_1 \sim t\int_0^\infty \frac{e^{st-\sqrt{s/k}\,x}}{s}\,ds = I_2$. To find I_2, set up contour integration using:

$$\tilde{C} = C_0 + C_1^+ + C_1^- + C_2^+ + C_2^- + C_3$$



$\underline{\tilde{C}}$: $\int_{\tilde{C}} \frac{1}{s} e^{st - \sqrt{\frac{s}{k}}x} \, ds = 0$

$\underline{C_1^+}$: $s = re^{i\theta}$ $\quad \theta \in (\alpha, \pi)$ $\left<\begin{array}{ll} \theta \in (\alpha, \frac{\pi}{2}) & r\cos\theta < c \\ \theta \in (\frac{\pi}{2}, \pi) & \cos\theta < 0 \end{array}\right.$

$$\left| \int_{c-i\infty}^{c+i\infty} t \frac{e^{st - \sqrt{\frac{s}{k}}x}}{s} ds \right| \leq t \int_\alpha^\pi \frac{e^{tr\cos\theta - \sqrt{\frac{r}{k}} \overbrace{\cos\frac{\theta}{2}}^{>0} x}}{\cancel{r}} \cancel{r} \, d\theta$$

$$\leq t \int_0^{\pi/2} e^{tc - \sqrt{r/k} \cos\frac{\pi}{2} x} \, d\theta$$

$$+ t \int_{\pi/2}^{\pi} e^{tr\overbrace{\cos\theta}^{<0} - \sqrt{r/k} \overbrace{\cos\frac{\theta}{2}}^{>0}} \, d\theta$$

$$\longrightarrow 0 \quad \text{as} \quad r \to \infty$$

$\underline{C_1^-}$: Similar to C_1^+ but $\theta \in (-\alpha, -\pi)$ $\left<\begin{array}{ll} \theta \in (-\alpha, -\frac{\pi}{2}) & r\cos\theta < c \\ \theta \in (-\frac{\pi}{2}, -\pi) & \cos\theta < 0 \end{array}\right.$

$\underline{C_3}$: $s = \varepsilon e^{i\theta}$ $\quad \theta \in (-\pi, \pi)$

$$t \int_\pi^{-\pi} \frac{e^{\varepsilon e^{i\theta} - \sqrt{\frac{\varepsilon}{k}} e^{i\frac{\theta}{2}}}}{\varepsilon e^{i\theta}} \varepsilon i e^{i\theta} d\theta = -it \int_{-\pi}^{\pi} e^{\varepsilon e^{i\theta} - \sqrt{\frac{\varepsilon}{k}} e^{i\frac{\theta}{2}}} d\theta$$

$$\longrightarrow -it \int_{-\pi}^{\pi} d\theta \quad \text{as} \quad \varepsilon \to 0$$

$$= -2\pi t i$$

$\underline{C_2^+}$: $s = r e^{i\pi}$ $\quad r : \infty \longrightarrow 0$

$$\int_{C_2^+} \frac{e^{st - \sqrt{\frac{s}{K}}x}}{s} ds = \int_\infty^0 \frac{e^{-rt - \sqrt{\frac{r}{K}} e^{i\frac{\pi}{2}} x}}{r e^{i\pi}} e^{i\pi} dr$$

Let $\beta = x\sqrt{\frac{r}{K}}$

$$= - \int_0^\infty \frac{1}{r} e^{-rt - i\sqrt{r/K}\, x} dr$$

$$= - \int_0^\infty \frac{1}{r} e^{-rt} (\cos\beta - i \sin\beta)\, dr$$

$\underline{C_2^-}$: $s = r e^{-i\pi}$ $\quad r : 0 \rightarrow \infty$

$$\int_0^\infty \frac{1}{r} e^{-rt - \sqrt{\frac{r}{K}} e^{-i\frac{\pi}{2}} x} dr = \int_0^\infty \frac{1}{r} e^{-rt} (\cos\beta + i \sin\beta)\, dr$$

$$C_2^+ + C_2^- = 2i \int_0^\infty \frac{e^{-rt}}{r} \sin\beta \, dr$$

$$\sin\beta \sim \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} \beta^{2n+1}$$

$$\sin\left(\sqrt{\frac{r}{K}}\, x\right) \sim \sum_{n=0}^\infty \underbrace{\frac{(-1)^n \left(\frac{x}{\sqrt{K}}\right)^{2n+1}}{(2n+1)!}}_{a_n} r^{n+1/2}$$

$$\underbrace{\frac{\sin\left(\sqrt{r/K}\, x\right)}{r}}_{*} \sim r^{-1/2} \sum_{n=0}^\infty a_n r^n \qquad \begin{array}{l} \alpha = -1/2 \\ \beta = 1 \end{array}$$

$$2ti \int_0^\infty * \, e^{-rt} dr \sim 2ti \frac{a_0 \Gamma(\alpha + \beta n + 1)}{t^{\alpha + \beta n + 1}} = 2ti \frac{x}{\sqrt{K}} \frac{\Gamma(3/2)}{t^{3/2}}$$

Watson's lemma

As $r \to \infty$ and $\varepsilon \to 0$ we get:

$$u \sim \frac{1}{\pi i} I_2 = \int_{C_0} \sim -\int_{C_2^+} - \int_{C_2^-} - \int_{C_3}$$

$$= \frac{t}{\pi i}\left(-2i \frac{x\,\Gamma(3/2)}{\sqrt{K}\; t^{3/2}} + 2\pi i\right)$$

$$= 2t - \frac{2\;x\,\Gamma(3/2)}{\pi\sqrt{Kt}}$$

Problem 3

$u_t = \nabla(K \nabla u)$ $\qquad K = A + f\left(\frac{x}{\varepsilon^2}\right) + g\left(\frac{y}{\varepsilon}\right)$

$u|_{t=0} = u(x,y)$ $\qquad u = u(\omega_1, z_2, x, y, t)$

$\omega = (\omega_1, \omega_2) = \left(\frac{x}{\varepsilon^2}, \frac{y}{\varepsilon^2}\right)$

$z = (z_1, z_2) = \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$

$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots \qquad \varepsilon \ll 1$

$$\frac{\partial}{\partial t}(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots) = \left(\frac{1}{\varepsilon^2}\nabla_\omega + \frac{1}{\varepsilon}\nabla_z + \nabla_x\right)\cdot\left(K\left(\frac{1}{\varepsilon^2}\nabla_\omega + \frac{1}{\varepsilon}\nabla_z + \nabla_x\right)(u_0 + \varepsilon u_1 + \dots\right)$$

Let $\quad L = \nabla_\omega (K \nabla_\omega)$

$O\left(\frac{1}{\varepsilon^4}\right)$: $\quad L u_0 = 0 \quad \Rightarrow \quad u_0(z_2, x, y, t)$

$O\left(\frac{1}{\varepsilon^3}\right)$: $\quad L u_1 + \nabla_\omega (K \nabla_z u_0) + \cancel{\nabla_z (K \nabla_\omega u_0)}^{\,0} = 0$

$\hookrightarrow K_{\omega_1} u_{0 z_1} + K_{\omega_2} u_{0 z_2} = 0$

$\Rightarrow u_1(\omega_1, z_2, x, y, t) = u_1(z_2, x, y, t)$

$O\left(\frac{1}{\varepsilon^2}\right)$: $\quad \underbrace{L u_2 + \nabla_\omega (K \nabla_x u_0)}_{*} + \underbrace{\nabla_z (\cancel{K \nabla_z u_0})^{\,0}}_{\text{by FA} \;\; **} + \underbrace{\nabla_\omega (K \nabla_z u_1)}_{\text{total deriv F.A.} \;\; ***} + \cancel{\nabla_z (K \nabla_\omega u_1)}^{\,0} + \cancel{\nabla_x (K \nabla_\omega u_0)}^{\,0} = 0$

$** \Rightarrow u_0 = u_0(x, y, t)$

$*** \Rightarrow u_{0 z_2} = 0 \quad$ so $\quad *** = 0$

$* = L u_2 + \nabla_\omega (K \nabla_x u_0) = 0 \quad \Rightarrow \quad L u_2 = -\nabla_\omega (K \nabla_x u_0)$

$\Downarrow$

$\nabla_\omega \cdot (K \nabla_\omega u_2) + \nabla_\omega \cdot (K \nabla_x u_0) = 0$

$\nabla_\omega \cdot (K \nabla_\omega u_2 + K \nabla_x u_0) = 0$

$K \nabla_\omega u_2 = -K \nabla_x u_0 + \vec{B}(x, y, t, z)$

$\Rightarrow u_2 = \theta_2(\omega_1, z_2, x, y, t)\frac{\partial u_0}{\partial x} + \bar{u}_1(z_2, x, y, t)$

$u_{2\omega_1} = -u_{0x} + \dfrac{u_{0x} \langle K \rangle_\omega^h}{K}$

$K u_{2\omega_1} = -u_{0x} + u_{0x} \langle K \rangle_\omega^h$

$K u_{2\omega_1} = (\langle K \rangle_\omega^h - 1) u_{0x}$

$$L u_2 = -\nabla_w (K \nabla_x u_0)$$

$$\frac{\partial}{\partial \omega_1}\left(K \frac{\partial}{\partial \omega_1} u_2\right) = \frac{\partial}{\partial \omega_1}\left(-K \frac{\partial}{\partial \omega_1}\theta\right) \frac{\partial u_0}{\partial x}$$

$$= -\frac{\partial}{\partial \omega_1} K \frac{\partial u_0}{\partial x}$$

$$\frac{\partial}{\partial \omega_1}\left(-K \frac{\partial \theta}{\partial \omega_1}\right) = \frac{\partial}{\partial \omega_1} K$$

$$\theta_{\omega_1} = -1 + \frac{B(z_2, x, y, t)}{K}$$

$$K \theta_{\omega_1} = -K + \langle K \rangle_w^h$$

$$\Theta\left(\tfrac{1}{\varepsilon}\right): \quad L u_3 + \underbrace{\cancel{\nabla_w (K \nabla_x u_1)}}_{\text{tot deriv}} + \nabla_z (K \nabla_z u_1) + \cancel{\nabla_w (K \nabla_z u_2)}^{0} + \nabla_z (K \nabla_w u_2)$$

$$+ \cancel{\nabla_x (K \nabla_w u_1)} + \nabla_z (K \nabla_x u_0) + \cancel{\nabla_x (K \nabla_z u_0)} = 0$$

$$0 = \langle L u_3 \rangle = -\nabla_z \left(\langle K(\omega_1, z_2) \rangle_w \nabla_z u_1 + \langle K \theta_{\omega_1} \rangle_w \frac{\partial u_0}{\partial x} + \langle K \rangle_w \nabla_x u_0 \right)$$

$$\frac{\partial}{\partial z_2}\left(\langle K \rangle_w \frac{\partial u_1}{\partial z_2}\right) = -\frac{\partial}{\partial z_2}\left(\langle K \rangle_w \frac{\partial u_0}{\partial y}\right)$$

$$\nabla_z \left(\langle K \theta_{\omega_1} \rangle_w \frac{\partial u_0}{\partial x}\right) = -\frac{\partial}{\partial z_1}\left(\langle K \theta_{\omega_1} \rangle_w \frac{\partial u_0}{\partial x}\right) = 0$$

$$u_1(z_2, x, y, t) = \phi(z_2) \frac{\partial u_0}{\partial y} + \overline{u}_2$$

$$\langle K \rangle_w \frac{\partial u_1}{\partial z_1} = \langle K \rangle_w \frac{\partial \phi}{\partial z_2} = -\langle K \rangle_w + c(x, y, t)$$

$$\frac{\partial u_1}{\partial z_1} = -1 + \frac{c}{\langle K \rangle_w}$$

$$0 = -1 + c \frac{1}{L} \int_0^L dz_1 \frac{1}{\langle K \rangle_w} = -1 + c \langle \langle K \rangle_w \rangle_z^h$$

$$c = \langle \langle K \rangle_w \rangle_z^h$$

Now: $$u_{z_{w_1}} = -u_{0x} + u_{0x} \langle K \rangle_w^h$$

$$K u_{z_{w_1}} = \left(-K + \langle K \rangle_w^h\right) u_{0x}$$

$$\langle K \rangle_w u_{1_{z_1}} = -\langle K \rangle + \langle\langle K \rangle_w \rangle_z^h$$

$O(1)$: $$L u_1 + \nabla_w \cancelto{0}{(K\nabla_x u_2)} + \nabla_z \cancelto{0}{(K\nabla_w u_2)} + \nabla_w \cancelto{0}{(K\nabla_z u_3)} + \nabla_z \cancelto{0}{(K\nabla_w u_3)}$$
$$+ \nabla_x (K \nabla_w u_2) + \nabla_z \cancelto{0}{(K\nabla_x u_1)} + \nabla_x (K \nabla_z u_1) + \nabla_x (K \nabla_x u_0) = \frac{\partial u_0}{\partial t}$$

$$0 = \frac{\partial u_0}{\partial t} - \nabla_x \left(\langle K \nabla_w u_2 \rangle_{w,t}\right) - \nabla_x \left(\langle K \nabla_z u_1 \rangle_{w,z}\right) - \nabla_x \left(\langle K \rangle_{w,z} u_0\right)$$

1 2 3

$$K \nabla_z u_1 = K(w_1, z_2)\left(0, \frac{d\phi}{dz_2} \frac{\partial u_0}{\partial y}\right)$$

$$\langle\langle K \frac{d\phi}{dz_1} \rangle\rangle_{w,z} \frac{\partial^2 u_0}{\partial y^2} = \langle\langle -(-K + \langle\langle K \rangle_w \rangle_z^h) \rangle\rangle \frac{\partial^2 u_0}{\partial t^2}$$
$$= \left(-\langle\langle K \rangle_w \rangle_z^h + \langle\langle K \rangle_w \rangle_z\right) \frac{\partial^2 u_0}{\partial t^2}$$

$$-\nabla_x \left(\langle\langle K \rangle_w \rangle_z \; \nabla_x u_0\right) = \langle\langle K \rangle_w \rangle_z \left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right)$$

$$\nabla_x (K \nabla_w u_2) = \langle\langle \nabla_x \left((-K + \langle K \rangle_w^h) \frac{\partial u_0}{\partial x}, 0\right) \rangle\rangle$$
$$= -\langle\langle K \rangle_w \rangle_z \frac{\partial^2 u_0}{\partial x^2} + \langle\langle K \rangle_w^h \rangle_z \frac{\partial^2 u_0}{\partial x^2}$$

$$\frac{\partial u_0}{\partial t} = \langle\langle K \rangle_w^h \rangle_z \frac{\partial^2 u_0}{\partial x^2} + \langle\langle K \rangle_w \rangle_z^h \frac{\partial^2 u_0}{\partial y^2}$$

Problem 4

Jan 2015

$$\varepsilon z^3 + z^2 = \varepsilon \cos z \qquad (1)$$

$$\underset{①}{\varepsilon z^3} + \underset{②}{z^2} \approx \underset{③}{\varepsilon}\Big(1 - \underset{④}{\tfrac{z^2}{2}}\Big) \qquad (2)$$

$$z_0 = \sum_{n=1}^{\infty} a_n \varepsilon^{n/2}$$

Plug z_0 into (2):

$$\varepsilon\left(a_1 \varepsilon^{1/2} + a_2\varepsilon + a_3 \varepsilon^{3/2} + \dots\right)^3 + \left(a_1\varepsilon^{1/2} + a_2\varepsilon + a_3\varepsilon^{3/2} + \dots\right)^2$$
$$= \varepsilon\left(1 - \tfrac{1}{2}\left(a_1\varepsilon^{1/2} + a_2\varepsilon + a_3\varepsilon^{3/2} + \dots\right)^2\right)$$

$O(\varepsilon)$: $a_1^2 = 1$

$a_1 = \pm 1$

$O(\varepsilon^{3/2})$: $2a_1 a_2 = 0 \Rightarrow a_2 = 0$

$O(\varepsilon^2)$: $\cancel{a_2^2} + 2a_1 a_3 - \frac{1}{2}a_1^2 = 0 \Rightarrow a_3 = \frac{a_1}{4}$

$O(\varepsilon^{5/2})$: $a_1^3 + \cancel{2a_2a_3} + 2a_1a_4 = \cancel{-a_1a_2}$

$$a_4 = -\frac{a_1^2}{2} = -\tfrac{1}{2}$$

$$z_1 \sim \varepsilon^{1/2} + \tfrac{1}{4}\varepsilon^{3/2} - \tfrac{1}{2}\varepsilon^2$$
as $\varepsilon \to 0$
$$z_2 \sim -\varepsilon^{1/2} - \tfrac{1}{4}\varepsilon^{3/2} - \tfrac{1}{2}\varepsilon^2$$

Other roots can be found using $z = \varepsilon^{-a}x$, subbing this into (2), balancing either ① and ② or ① and ③, then solving for x roots like we just did. Reusing the fact that $z = \varepsilon^{-\alpha}x$ gives the other z roots. Should get 4 total.

Problem 5

Jan 2015

$$\phi_t + \phi_{xxx} - \varepsilon \phi_{xx} = 0 \qquad 0 < \varepsilon \ll 1 \qquad (1)$$

a) $\phi(x,t) = e^{i(kx-\omega t)}$

$\phi_t = -i\omega e^{i(kx-\omega t)}$

$\phi_x = ik e^{i(kx-\omega t)}$

Sub into (1):

$$-i\omega \phi + (ik)^3 \phi - \varepsilon (ik)^2 \phi = 0$$

$$-\omega - k^3 - \varepsilon i k^2 = 0$$

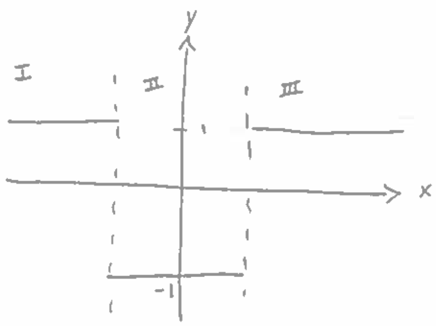
$$\boxed{\omega = W(k) = -k^3 - \varepsilon i k^2}$$

b) $\phi(x,t) = \int_{-\infty}^{\infty} F(k) e^{i(kx + k^3 t + \varepsilon i k^2 t)}$

Problem 6

$$\varepsilon y'' - U(x)y = \lambda y \qquad \Rightarrow \qquad y'' = \frac{\lambda + U(x)}{\varepsilon} y$$

$$U(x) = \begin{cases} 1 & |x| > 1 \\ -1 & |x| < -1 \end{cases}$$



a) To ensure a bounded soln we should assume

$$\max Q(x) > E > \min Q(x)$$

$$\max U(x) > -\lambda > \min U(x)$$

$$-1 < \lambda < 1$$

Region I: $\displaystyle y_1'' = \underbrace{\frac{\lambda+1}{\varepsilon}}_{>0} y_1$

$$y_1 = A_1 e^{\sqrt{\frac{\lambda+1}{\varepsilon}}x} + B_1 e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}x}$$

$$y_1' = A_1 \sqrt{\frac{\lambda+1}{\varepsilon}} e^{\sqrt{\frac{\lambda+1}{\varepsilon}}x} - B_1 \sqrt{\frac{\lambda+1}{\varepsilon}} e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}x}$$

Region II: $\displaystyle y_2'' = \underset{<0}{\frac{\lambda-1}{\varepsilon}} y_2$

$$y_2 = A_2 \sin\left(\sqrt{\frac{\lambda-1}{\varepsilon}}x\right) + B_2 \cos\left(\sqrt{\frac{\lambda-1}{\varepsilon}}x\right)$$

$$y_2' = A_2 \sqrt{\frac{\lambda-1}{\varepsilon}} \cos\left(\sqrt{\frac{\lambda-1}{\varepsilon}}x\right) - B_2 \sqrt{\frac{\lambda-1}{\varepsilon}} \sin\left(\sqrt{\frac{\lambda-1}{\varepsilon}}x\right)$$

Region III: $y_3'' = \underbrace{\frac{\lambda+1}{\varepsilon}}_{>0} y_3$

$$y_3 = A_3 e^{\sqrt{\frac{\lambda+1}{\varepsilon}}\,x} + B_3 e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}\,x}$$

$$y_3' = A_3\sqrt{\frac{\lambda+1}{\varepsilon}}\, e^{\sqrt{\frac{\lambda+1}{\varepsilon}}\,x} - B_3\sqrt{\frac{\lambda+1}{\varepsilon}}\, e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}\,x}$$

Now we have 6 unknown coeff's, but the patching we're about to do only yields 4 eqns. However, if we impose the condition $y \to 0$ as $|x| \to \infty$, we find $B_1 = 0$ and $A_3 = 0$ and we have a solvable system.

Patching:

$$y_1(-1) = y_2(-1)$$

$$A_1 e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}} = -A_2 \sin\left(\sqrt{\frac{\lambda-1}{\varepsilon}}\right) + B_2 \cos\left(\sqrt{\frac{\lambda-1}{\varepsilon}}\right)$$

$$y_1'(-1) = y_2'(-1)$$

$$A_1\sqrt{\frac{\lambda+1}{\cancel{\varepsilon}}}\, e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}} = A_2\sqrt{\frac{\lambda-1}{\cancel{\varepsilon}}}\cos\left(\sqrt{\frac{\lambda-1}{\varepsilon}}\right) + B_2\sqrt{\frac{\lambda-1}{\cancel{\varepsilon}}}\sin\left(\sqrt{\frac{\lambda-1}{\varepsilon}}\right)$$

$$y_2(1) = y_3(1)$$

$$A_2 \sin\sqrt{\frac{\lambda-1}{\varepsilon}} + B_2\cos\sqrt{\frac{\lambda-1}{\varepsilon}} = B_3 e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}}$$

$$A_2\sqrt{\lambda-1}\cos\left(\sqrt{\frac{\lambda-1}{\varepsilon}}\right) - B_2\sqrt{\lambda-1}\sin\left(\sqrt{\frac{\lambda-1}{\varepsilon}}\right) = -B_3\sqrt{\lambda+1}\, e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}}$$

$\uparrow$

$$y_2'(1) = y_3'(1)$$

$$\underbrace{\begin{pmatrix} e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}} & \sin\left(\sqrt{\frac{\lambda-1}{\varepsilon}}\right) & -\cos\left(\sqrt{\frac{\lambda-1}{\varepsilon}}\right) & 0 \\ \sqrt{\lambda+1}\, e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}} & -\sqrt{\lambda-1}\cos\left(\sqrt{\frac{\lambda-1}{\varepsilon}}\right) & -\sqrt{\lambda-1}\,\sin\left(\sqrt{\frac{\lambda-1}{\varepsilon}}\right) & 0 \\ 0 & \sin\left(\sqrt{\frac{\lambda-1}{\varepsilon}}\right) & \cos\left(\sqrt{\frac{\lambda-1}{\varepsilon}}\right) & -e^{\sqrt{\frac{\lambda+1}{\varepsilon}}} \\ 0 & \sqrt{\lambda-1}\cos\sqrt{\frac{\lambda-1}{\varepsilon}} & -\sqrt{\lambda-1}\,\sin\left(\sqrt{\frac{\lambda-1}{\varepsilon}}\right) & \sqrt{\lambda+1}\, e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}} \end{pmatrix}}_{M} \begin{pmatrix} A_1 \\ A_2 \\ B_2 \\ B_3 \end{pmatrix} = \vec{0}$$

To ensure we continue to have a non trivial solution we want $\det(M) = 0$.

<u>To finish:</u>

1) Find $\det M$
2) Factor result
3) Set each result equal to zero
4) Solve each remaining eqn for λ.

b) To do this, I would need to finish part a. The result in this piece should be equivalent to what's found in the next section.

c) $\frac{1}{\sqrt{\varepsilon}}\int_A^B \sqrt{E-Q}\, dx \sim (n+\frac{1}{2})\pi + O(\varepsilon)$ as $\varepsilon \to 0$ for $n \in \mathbb{N}$

$$\Rightarrow \frac{1}{\sqrt{\varepsilon}}\int_{-1}^{1} \sqrt{-\lambda-(-1)}\, dx = (n+\tfrac{1}{2})\pi$$

$$\sqrt{\frac{1-\lambda}{\varepsilon}}\ \underbrace{\int_{-1}^{1} dx}_{2} = (n+\tfrac{1}{2})\pi$$

$$1-\lambda = \frac{\varepsilon (n+\frac{1}{2})^2 \pi^2}{4}$$

$$\boxed{\lambda = 1 - \frac{\varepsilon}{4}(n+\tfrac{1}{2})^2 \pi^2}$$

Note this was only the bounded case. A full analysis of the problem would involve finding y_i, $i=1,2,3$, for the $\lambda>1$ and $\lambda<-1$ cases as well. In these cases, our solution would be unbounded. We should continue to use the assumption that $y \to 0$ as $|x| \to \infty$ so that the resulting system from patching can be solved in the same way described in part a.

Problem #7

Jan 2015

a) Let $\Omega \subset \mathbb{C}$, $x_0 \in \Omega$ and f_n be a sequence of funcs.

Ptws: We say $f_n \to f$ conv ptws if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ st $\forall n > N$ we get $|f_n(x_0) - f(x_0)| < \varepsilon$.

Note: We are only concerned with the fixed pt x_0 and N can vary as ε changes.

Asymp: Fix N. We say $f_N \to f$ conv asymp'ly if $\forall \varepsilon > 0\ \exists \delta > 0$ st $|x - x_0| < \delta \Rightarrow |f_N(x) - f(x)| < \varepsilon$.

Note: We are only concerned w/ the nbhd of x_0, not x_0 itself and N is fixed.

Ex: In each example, I'll use Ratio test to show conv.

i) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \left| \frac{x}{n+1} \right| \xrightarrow[n\to\infty]{x \text{ fixed}} 0 \quad \therefore \text{ ptws conv}$$

$$\xrightarrow[n \text{ fixed}]{x\to\infty} \infty \quad \therefore \text{ not asymp conv}$$

$\sum_{n=0}^{\infty} \frac{n!}{x^n}$

$$\left| \frac{(n+1)!}{x^{n+1}} \cdot \frac{x^n}{n!} \right| = \left| \frac{n+1}{x} \right| \xrightarrow[n\to\infty]{x \text{ fixed}} \infty \quad \therefore \text{ not ptw conv}$$

$$\xrightarrow[n \text{ fixed}]{x\to\infty} 0 \quad \therefore \text{ asymp conv}$$

b) Asymptotic expansions are not unique so the mapping is not one to one. However, you can say a function is asymptotic to itself (though that is not necessarilly helpful) so it is an onto mapping. Clearly, this means the mapping is not bijective.

c) Sps $f \sim \tilde{f}$ and $g \sim \tilde{g}$ as $z \to z_0$. We can say $f + g \sim \tilde{f} + \tilde{g}$ as $z \to z_0$ so $\sim$ is distributive. It's important to notice that both asymptotic relations occur as $z \to z_0$, z cannot approach different values for $f \sim \tilde{f}$ and $g \sim \tilde{g}$ or we won't have distributivity.