Spring 2016 Scientific Computation Comprehensive Examination

Answer 5 questions of your choice explaining all steps that lead to a solution. Partial credit will be awarded for presenting a viable solution strategy. No credit will be given to computations presented without explanation of the approach.

1. Consider the following $n \times n$ matrix:

$$A = n^2 \begin{bmatrix} -2 & 1 & 0 & \dots & 0 & 0 & 1 \\ 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & -2 & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 & -2 \end{bmatrix}.$$

   a. What relation exists between A and applying $D = \frac{d^2}{dx^2}$ to periodic functions?

   b. What are the eigenvalues λ_k and eigenvectors v_k of A?

   c. Apply the forward Euler method to system $y'(t) = Ay(t)$. Determine the step size Δt that ensures absolute stability.

2. Consider the iteration $x_{k+1} = x_k + \alpha_k r_k$ to solve the linear system $Ax = b$ starting from initial approximation x_0, with $r_k = b - A x_k$, α_k a scalar to be determined, and A symmetric positive definite with minimum, maximum eigenvalues $\lambda_{\min}$, $\lambda_{\max}$.

   a. Find α_k that minimizes $\|e_{k+1}\|_A^2 = e_{k+1}^T A e_{k+1}$, with $e_{k+1} = x - x_{k+1}$.

   b. Show that the error satisfies $\|e_k\|_A^2 \leqslant (1-w)^k \, \|e_0\|_A^2$, with $w = \lambda_{\min}/\lambda_{\max}$.

3. Find the best approximations to $f(x) = \sin(x)$ by a linear function $u(x) = \lambda x$ on the interval $[0, \frac{\pi}{2}]$ in (a) the maximum norm, and (b) the Euclidean norm.

4. Construct a fourth-order accurate approximation formula of $f'(x_0)$, based on values of $f: \mathbb{R} \to \mathbb{R}$ at points $x_i = x_0 + ih$, $i \in \mathbb{Z}$. Provide an estimate of the step size h that provides smallest relative error of the approximation in double precision floating point computation. Assume $f \in C^\infty(\mathbb{R})$.

5. Determine the end conditions for a cubic spline interpolation $S(x)$ of data $\mathcal{D} = \{(x_i, y_i = f(x_i)) \mid i = 0, \dots, n\}$, $f \in C^2(\mathbb{R})$, that minimize

$$I(S) = \int_a^b [S''(x)]^2 \, dx.$$

6. Consider $f\colon [0,h] \to \mathbb{R}$ an analytic function, $h > 0$, and the quadrature formula

$$\int_0^h \sqrt{x}\, f(x)\, \mathrm{d}x = A\, f(ah) + B\, f(bh) + C h^p.$$

a. Determine a, b, A, B that correspond to two-point Gauss-Legendre quadrature

$$\int_{-1}^{+1} F(x)\, \mathrm{d}x \cong F\left(-\frac{1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right).$$

Present possible deficiencies of this approach (use a counter-example).

b. Determine alternative values a, b, A, B such that the order of the quadrature method p is as high as possible. Compute C for this case.

Problem 1

a) Consider the ODE BVP:

$$\begin{cases} Du(x) = f(x) & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases}$$

If we discretize with n+1 nodes st $x_i = ih$ for $i = 0, 1, \dots, n$ where $h = \frac{1}{n}$ then $x_0 = 0$ and $x_n = 1$.

Let $u_i = u(x_i)$ and $\vec{u} = [u_1 \; u_2 \; \dots \; u_{n-1}]$

$$\Rightarrow \quad Du_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \qquad i = 1, 2, \dots, n-1$$

Let $\vec{f} = [f_1 \; f_2 \; \dots \; f_{n-1}]^T$ and $\vec{g} = [f(0) \; 0 \dots 0 \; f(1)]^T$

then our system is:

$$A\vec{u} = \vec{f}.$$

b) Since $\sin(\beta x)$ is an eigenfunction of D, consider its discretized counterpart:

$$\vec{v}_k = \left[\sin\left(\frac{k\pi}{n}\right) \quad \sin\left(\frac{2k\pi}{n}\right) \quad \dots \quad \sin\left(\frac{(n-1)k\pi}{n}\right)\right]^T$$

Plugging into $A\vec{v}_k = \lambda_k v_k$, we get:

$$\alpha = \frac{k\pi}{n}$$

$$h^2 \lambda_k \sin\left(\frac{jk\pi}{n}\right) = \sin\left(\frac{(j-1)k\pi}{n}\right) - 2\sin\left(\frac{jk\pi}{n}\right) + \sin\left(\frac{(j+1)k\pi}{n}\right)$$

$$= \sin\left(\frac{jk\pi}{n} - \frac{k\pi}{n}\right) - 2\sin\left(\frac{jk\pi}{n}\right) + \sin\left(\frac{jk\pi}{n} + \frac{k\pi}{n}\right)$$

$$= \sin(j\alpha)\cos(\alpha) - \cancel{\sin(\alpha)\cos(j\alpha)} - 2\sin(j\alpha) + \sin(j\alpha)\cos(\alpha) + \cancel{\sin(\alpha)\cos(j\alpha)}$$

$$= 2\sin(j\alpha)(\cos(\alpha) - 1)$$

$$\Rightarrow \lambda_k = 2\left(\cos\left(\frac{jk\pi}{n}\right) - 1\right)\frac{1}{h^2}$$

c) Recall the forward Euler method: $u^{n+1} = u^n + k A u^n$

To show stability assume $u'(x) = f(x) = \lambda u$

$\Rightarrow A u^n = \lambda u^n$

$\Rightarrow u^{n+1} = u^n + k\lambda u^n = (1+k\lambda) u^n = (1+k\lambda)^{n+1} u^0$

For convergence we need $|1+k\lambda| < 1$

$$-1 < 1 + k\lambda < 1$$

$$-2 < k\lambda < 0$$

$$-2 < \frac{k}{h^2}\left(2\left(\cos\left(\frac{j\pi}{n}\right) - 1\right)\right) < 0$$

$$-1 < \frac{k}{h^2}\left(\cos\left(\frac{j\pi}{n}\right) - 1\right) < 0$$

$$1 > \underbrace{\frac{k}{h^2}}_{<1} \underbrace{\left(1 - \cos\left(\frac{j\pi}{n}\right)\right)}_{\leq 1} > 0$$

$\Leftarrow$

$\Rightarrow h^2 > k$

Problem 2 Jan 2016

a) $\|e_{k+1}\|_A = (x - x_{k+1})^T A (x - x_{k+1}) = x^T A x - 2x_{k+1}^T A x + x_{k+1}^T A x_{k+1}$

$= x^T b - 2x_{k+1}^T b + x_{k+1}^T A x_{k+1}$

$= x^T b - 2(x_k + \alpha_k r_k)^T b - (x_k + \alpha_k r_k)^T A (x_k + \alpha_k r_k)$

$= x^T b - 2x_k b^T - 2\alpha_k r_k^T b + x_k^T A x_k + 2\alpha_k x_k^T A r_k + \alpha_k^2 r_k^T A r_k$

$= p(\alpha_k)$

Want to minimize

$$\frac{\partial p}{\partial \alpha} = -2r_k^T b + 2x_k^T A r_k + 2\alpha_k r_k^T A r_k = 0$$

$$\Rightarrow \quad \alpha_k = \frac{r_k^T b - x_k^T A r_k}{r_k^T A r_k}$$

b) $\|e_k\|_A^2 \le (1-\omega)\|e_{k-1}\|_A^2$

$e_k^T A e_k = (x - x_k)^T A (x - x_k) = \|e_{k-1}\|_A^2 - \alpha_k r_k^T A e_{k-1}$

?

Try diagnolizing A?

Note:

$\lambda_{max}^2 = \|A\|_2^2$

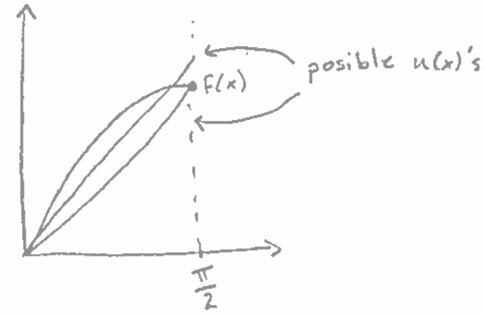
$\frac{1}{\lambda_{min}^2} = \|A^{-1}\|_2^2$

Jan 2016

Problem 3

$f(x) = \sin x \qquad u(x) = \lambda x$

a)

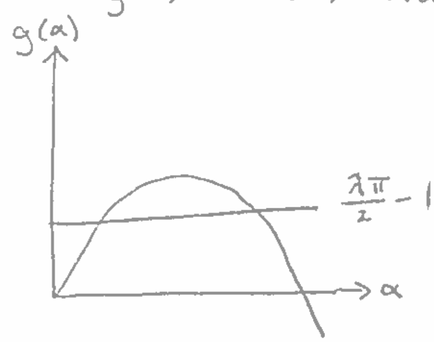


$\sin(x)$ is monotonically incr on $[0, \pi/2]$ so max error should occur at $x = \frac{\pi}{2}$.

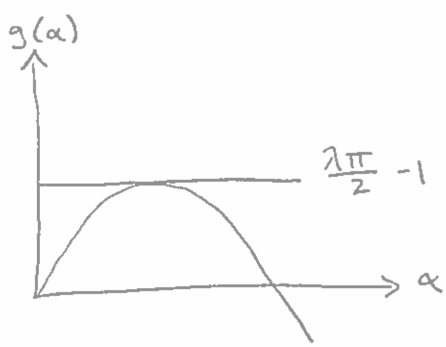
At $x = \frac{\pi}{2}$: $\text{error} = \sin(\frac{\pi}{2}) - \lambda(\frac{\pi}{2}) = 1 - \frac{\lambda\pi}{2} = E$

$\exists\ \alpha \in (0, \frac{\pi}{2})$: $\text{error} = \sin(\alpha) - \lambda\alpha = -E = \frac{\lambda\pi}{2} - 1$

Let $\underbrace{g(\alpha)}_{\text{error func}} = \sin(\alpha) - \lambda\alpha$



As prev stated, max error is $\left|1 - \frac{\lambda\pi}{2}\right|$ so the graph should have one soln thus $\frac{\lambda\pi}{2} - 1$ should occur at the peak.



$g'(\alpha) = \cos(\alpha) - \lambda = 0$

$\lambda = \cos\alpha$

$\alpha = \arccos(\lambda)$

Note: this means $\lambda \leq 1$ but checking $\lambda = 1$ makes it clear $\lambda \neq 1$ so $\lambda < 1$.

$g(\alpha) = g(\arccos(\lambda)) = \sin(\arccos(\lambda)) - \lambda \arccos(\lambda) = \frac{\lambda\pi}{2} - 1$

solve for λ

b) $E = \int_0^{\pi/2} (\sin(x) - \lambda x)^2 \, dx$ by defn of Euclidean norm

Want to minimize error so:

$$\frac{dE}{d\lambda} = \int_0^{\pi/2} 2(\sin(x) - \lambda x)(-x)\, dx = 0$$

$$\int_0^{\pi/2} (\lambda x^2 - x\sin x)\, dx = 0$$

$$\frac{\lambda}{3} x^3 \Big|_0^{\pi/2} - \int_0^{\pi/2} x \sin x \, dx = 0 \qquad \begin{array}{ll} u = x & dv = \sin x \, dx \\ du = dx & v = -\cos x \end{array}$$

$$\frac{\lambda}{3}\left(\frac{\pi}{2}\right)^3 + x\cos x \Big]_0^{\pi/2} - \int_0^{\pi/2} \cos x \, dx = 0$$

$$\frac{\lambda \pi^3}{24} - \sin x \Big|_0^{\pi/2} = 0$$

$$\frac{\lambda \pi^3}{24} = 1$$

$$\boxed{\lambda = \frac{24}{\pi^3}}$$

$f(x) = \sin(x)$ on $[0, \frac{\pi}{2}]$

$u(x) = \lambda x$

b) $E = \int_0^{\pi/2} (\sin(x) - \lambda x)^2 \, dx$

$$\frac{\partial E}{\partial \lambda} = 2 \int_0^{\pi/2} (\sin x - \lambda x)(-x) \, dx = 0$$

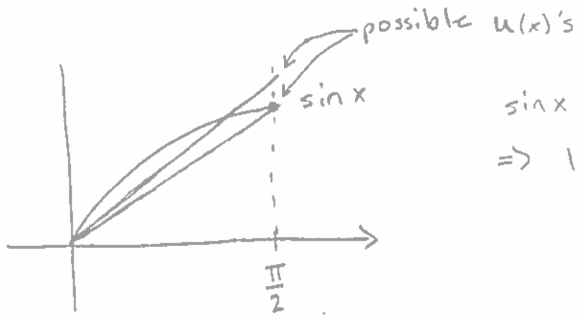
$$\Rightarrow \quad \int_0^{\pi/2} (\lambda x^2 - x\sin x) \, dx = 0$$

$$\frac{\lambda}{3}\left(\frac{\pi}{2}\right)^3 - 0 - \left(-x\cos x\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx\right) = 0$$

$$\frac{\lambda \pi^3}{24} + \cos x \Big|_0^{\pi/2} = 0$$

$$\frac{\lambda \pi^3}{24} - 1 = 0$$

$$\boxed{\lambda = \frac{24}{\pi^3}}$$

a) 

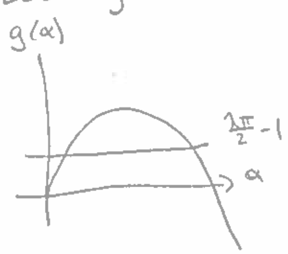
$\sin x$ is monotonically incr on $[0, \frac{\pi}{2}]$

$\Rightarrow$ largest error should occur at $x = \frac{\pi}{2}$

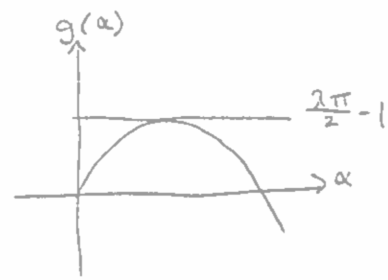
At $x = \frac{\pi}{2}$: $E = \sin\frac{\pi}{2} - \lambda\frac{\pi}{2} = 1 - \frac{\lambda\pi}{2}$

For $x \in (0, \frac{\pi}{2})$: $-E = \sin\alpha - \lambda\alpha = \frac{\lambda\pi}{2} - 1$

Let $g(\alpha) = \sin(\alpha) - \lambda\alpha$



Want this to have one soln. which only happens $\frac{\lambda\pi}{2} - 1$ occurs at the peak



Problem 4

Let $x_i = ih$ where $i \in \mathbb{Z}$ (note $x_0 = 0$) and assume

$$f'(x_0) = a f(x_{-2}) + b f(x_{-1}) + c f(x_0) + d f(x_1) + e f(x_2)$$

$f(x) = 1$: $\quad 0 = a + b + c + d + e$

$f(x) = x$: $\quad 1 = -2ha - hb + hd + 2he$

$f(x) = x^2$: $\quad 0 = 4h^2 a + h^2 b + h^2 d + 4h^2 e$

$f(x) = x^3$: $\quad 0 = -8h^3 a - h^3 b + h^3 d + 8h^3 e$

$f(x) = x^4$: $\quad 0 = 16h^4 a + h^4 b + h^4 d + 16h^4 e$

$$\begin{cases} (1) & a + b + c + d + e = 0 \\ (2) & -2ha - hb + hd + 2he = 1 \quad \Rightarrow -2a - b + d + 2e = \frac{1}{h} \\ (3) & 4a + b + d + 4e = 0 \\ (4) & -8a - b + d + 8e = 0 \\ (5) & 16a + b + d + 16e = 0 \end{cases}$$

(3) + (2), (4) − (2), (5) + (2)

$$\begin{cases} 2a + 2d + 6e = 1/h \\ -6a + 6e = -1/h \\ 14a + 2d + 18e = 1/h \end{cases} \rightarrow \begin{array}{l} 8a + 2d = 2/h \\ 32a + 2d = 4/h \end{array} \Rightarrow \begin{cases} -4a - d = -\frac{1}{h} \\ 16a + d = \frac{2}{h} \end{cases}$$

$$12a = \frac{1}{h}$$

$\underline{a = \frac{1}{12h}} \quad \Rightarrow \quad e = -\frac{1}{6h} + a = -\frac{2}{12h} + \frac{1}{12h} = \underline{-\frac{1}{12h} = e}$

$\cancel{2}\left(\frac{1}{12h}\right) + \cancel{2}d + \cancel{6}^{3}\left(-\frac{1}{12h}\right) = \frac{1}{h} \quad \cdot \frac{1}{2}$

$d = \frac{6}{12h} + \frac{3}{12h} - \frac{1}{12h} = \underline{\frac{2}{3h} = d}$

$b = -\frac{4}{12h} - \frac{2}{3h} + \frac{4}{12h} = \underline{-\frac{2}{3h} = b}$

$\frac{1}{12h} - \frac{2}{3h} + c + \frac{2}{3h} - \frac{1}{12h} = 0 \quad \Rightarrow \quad \underline{c = 0}$

$a = \frac{1}{12h}$ $b = -\frac{2}{3h}$ $c = 0$ $d = \frac{2}{3h}$ $e = \frac{1}{12h}$

$$f'(x_0) \approx \frac{1}{h}\left(\frac{1}{12} f(x_{-2}) - \frac{2}{3} f(x_{-1}) + \frac{2}{3} f(x_1) - \frac{1}{12} f(x_2)\right) = L(f,h)$$

Fourth Order Accurate? Let $f^{(n)}(0) = f^{(n)}$

$$\frac{2}{3}\left(f(h) = f + hf' + \frac{h^2}{2} f'' + \frac{h^3}{3!} f''' + \frac{h^4}{4!} f^{(4)} + \frac{h^5}{5!} f^{(5)} + \Theta(h^6)\right)$$

$$-\frac{2}{3}\left(f(-h) = f - hf' + \frac{h^2}{2} f'' - \frac{h^3}{3!} f''' + \frac{h^4}{4!} f^{(4)} - \frac{h^5}{5!} f^{(5)} + \Theta(h^6)\right)$$

$$* = \frac{4h}{3} f' + \frac{2h^3}{9} f''' + \frac{4h^5}{3 \cdot 5!} f^{(5)} + \Theta(h^7)$$

$$-\frac{1}{12}\left(f(2h) = f + 2hf' + 2h^2 f'' + \frac{4h^3}{3} f''' + \frac{2h^4}{3} f^{(4)} + \frac{4h^5}{15} f^{(5)} + \Theta(h^6)\right)$$

$$\frac{1}{12}\left(f(-2h) = f - 2hf' + 2h^2 f'' - \frac{4h^3}{3} f''' + \frac{2h^4}{3} f^{(4)} - \frac{4h^5}{15} f^{(5)} + \Theta(h^6)\right)$$

$$** = -\frac{h}{3} f' - \frac{2h^3}{9} f''' - \frac{2h^5}{3 \cdot 15} f^{(5)} + \Theta(h^7)$$

$$\frac{1}{h}(* + **) = f' + \left(\frac{4}{3 \cdot 5!} - \frac{2}{45}\right) h^4 f^{(5)} + \Theta(h^6)$$

$$= f' + \Theta(h^4) \qquad \frac{1}{90} - \frac{4}{90} = -\frac{1}{30}$$

So it is 4^{th} order accurate

Relative Error

$$R = \frac{f'(x_0) - L(f,h)}{f'(x_0)} \approx \frac{f'(x_0) - \left(f'(x_0) - \frac{1}{30} h^4 f^{(5)}(x_0) + \Theta(h^6)\right)}{f'(x_0)} \qquad \frac{\Theta(h^7)}{h}$$

$$= \frac{\frac{h^4}{30} f^{(5)}(x_0) + \frac{\varepsilon}{h}}{f'(x_0)}$$

Minimize R:

$$0 = \frac{dR}{dh} = \frac{\frac{2}{15} h^3 f^{(5)}(x_0) - \frac{\varepsilon}{h^2}}{f'(x_0)} \Rightarrow h = \left(\frac{15\varepsilon}{2 f^{(5)}(x_0)}\right)^{1/5}$$

Problem 4

$$f'(x_0) = af(x_{-2}) + bf(x_{-1}) + cf(x_0) + df(x_1) + ef(x_2)$$

$\underline{f(x)=1:}$ $\quad 0 = a+b+c+d+e$

$\underline{f(x)=x-x_0:}$ $\quad 1 = a(\cancel{x_0}-2h\cancel{-x_0}) - bh + c\cdot(0) + dh + e(2h)$

$\underline{f(x)=(x-x_0)^2:}$ $\quad 0 = 4h^2a + h^2b + h^2d + 4h^2e$

$\underline{f'(x)=(x-x_0)^3:}$ $\quad 0 = -8h^3a - h^3b + h^3d - 8h^3e$

$\underline{f(x)=(x-x_0)^4:}$ $\quad 0 = 16h^4a + h^4b + h^4d + 16h^4e$

$$\begin{cases} a + b + c + d + e = 0 \\ -2ha - hb + \quad hd + 2he = 1 \\ 4a + b + \quad d + 4e = 0 \\ -8a - b + \quad d + 8e = 0 \\ 16a + b + \quad d + 16e = 0 \end{cases}$$

$$\Rightarrow a = \frac{1}{12h} \qquad b = -\frac{2}{3h} \qquad c = 0 \qquad d = \frac{2}{3h} \qquad e = -\frac{1}{12h}$$

$$\Rightarrow f'(x_0) \approx \frac{1}{h}\left(\frac{1}{12}f(x_{-2}) - \frac{2}{3}f(x_{-1}) + \frac{2}{3}f(x_1) - \frac{1}{12}f(x_2)\right) \equiv L(f,h)$$

But is this order h^4? Do out Taylor series and get:

$$f'(x_0) = L(f,h) - \frac{1}{30}f^{(5)}(x_0)h^4 + O(h^5) \qquad \ast$$

Relative Error:

$$R = \frac{f'(x_0) - L(f,h)}{f'(x_0)} \underset{\ast}{\approx} \frac{f'(x_0) - \left(f'(x_0) + \frac{1}{30}f^{(5)}(x_0)h^4 + \frac{\varepsilon}{h}\right)}{f'(x_0)} = \frac{-\frac{1}{30}f^{(5)}(x_0)h^4 - \frac{\varepsilon}{h}}{f'(x_0)}$$

Need to minimize R:

$$0 = \frac{dR}{dh} = -\frac{2}{15}f^{(5)}(x_0)h^3 + \frac{\varepsilon}{h^2} \quad \Rightarrow \quad \boxed{h = \left(\frac{15\varepsilon}{2f^{(5)}(x_0)}\right)^{1/5}}$$

Jan 2016

Problem 5

$$I(s) = \int_a^b (s''(x))^2 \, dx$$

For minimization

$$\frac{dI}{ds} = \int_a^b 2s''(x)s'''(x)\,dx = 0$$

Recall: $S_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3$

$$s_i'(x) = b_i + 2c_i(x-x_i) + 3d_i(x-x_i)^2$$

$$s_i''(x) = 2c_i + 6d_i(x-x_i) \quad \Rightarrow \quad c_i = \frac{s_i(x_i)}{2}$$

$$s_i'''(x) = 6d_i$$

$$0 = 2\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} s_i''(x)\, s_i'''(x)\,dx$$

$$= 2\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (2c_i + 6d_i(x-x_i))(6d_i)\,dx$$

$$= 12\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (2c_i d_i + 6d_i^2(x-x_i))\,dx$$

$$= 12\sum_{i=0}^{n-1} \left(2c_i d_i(x_{i+1}-x_i) + 3d_i^2(x_{i+1}-x_i)^2\right)$$

$$= 12\sum_{i=0}^{n-1} d_i(x_{i+1}-x_i)(c_i + \underbrace{c_i + 3d_i(x_{i+1}-x_i)}_{\frac{s_i''(x_{i+1})}{2} = \frac{s_{i+1}''(x_{i+1})}{2} = c_{i+1}})$$

$$= 4\sum_{i=0}^{n-1} (-c_i + c_i + 3d_i(x_{i+1}-x_i))(c_i + c_{i+1})$$

$$= 4\sum_{i=0}^{n-1} (-c_i + c_{i+1})(c_i + c_{i+1})$$

$$= 4\sum_{i=0}^{n-1} c_{i+1}^2 - c_i^2 = 4(c_n^2 - c_0^2) = 0$$

telescoping series

Problem 5

Jan 2016

$$I(S) = \int_a^b [S''(x)^2]dx$$

$$\frac{dI}{dS} = \int_a^b 2S''(x)S'''(x)\,dx$$

$$= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} 2S_i'' S_i'''\,dx$$

Recall: $s(x) = a_j + b_j(x-x_j) + c_j(x-x_j)^2 + d_j(x-x_j)^3$

$$s'(x) = b_j + 2c_j(x-x_j) + 3d_j(x-x_j)^2$$

$$s''(x) = 2c_j + 6d_j(x-x_j)$$

$$s'''(x) = 6d_j$$

$$\Rightarrow \quad a_j = S_j(x_j) \quad b_j = S_j'(x_j) \quad c_j = \frac{S_j''(x_j)}{2} \quad d_j = \frac{S_j'''(x_j)}{6}$$

$$= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left(12c_id_i + 36d_i^2(x-x_i)\right)dx$$

$$= \sum_{i=0}^{n-1} \left(12c_id_ih_i + 18d_i^2h_i^2\right)$$

$$= \sum_{i=0}^{n-1} 4c_i(c_{i+1}-c_i) + 2(c_{i+1}-c_i)^2 = 0 \quad \text{for minimization}$$

$$\Rightarrow \quad \sum_{i=0}^{n-1} 2c_{i+1} - 2c_i = 0$$

$$\sum_{i=0}^{n-1} c_{i+1} - c_i = 0$$

$$c_n - c_0 = 0 \quad \because \text{telescoping series}$$

$$c_n = c_0$$

$$\frac{S_0''(x_0)}{2} = \frac{S_n''(x_n)}{2}$$

$$S_0''(x_0) = S_n''(x_n) \longleftarrow \text{BC's}$$

Problem 6

$$\int_0^h \sqrt{x}\, f(x)\, dx = A f(ah) + B f(bh) + C h^p$$

a) Ntf $y(x)$ st $y(0) = -1$ + $y(h) = 1$

$$y = \frac{2}{h} x - 1 \quad \Rightarrow \quad x = \frac{h}{2}(y+1)$$
$$dx = \frac{h}{2} dy$$

$$I(f) = \int_0^h \sqrt{x}\, f(x)\, dy = \int_{-1}^{1} \sqrt{\tfrac{h}{2}(y+1)}\; f\left(\tfrac{h}{2}(y+1)\right) \tfrac{h}{2}\, dy \approx F\left(-\tfrac{1}{\sqrt{3}}\right) + F\left(\tfrac{1}{\sqrt{3}}\right)$$

$$\Rightarrow F(y) = \frac{h}{2}\sqrt{\tfrac{h}{2}(y+1)}\; f\left(\tfrac{h}{2}(y+1)\right)$$

$$I(f) \approx \frac{h}{2}\sqrt{\tfrac{h}{2}\left(1-\tfrac{1}{\sqrt{3}}\right)}\; f\left(\tfrac{h}{2}\left(1-\tfrac{1}{\sqrt{3}}\right)\right) + \frac{h}{2}\sqrt{\tfrac{h}{2}\left(1+\tfrac{1}{\sqrt{3}}\right)}\; f\left(\tfrac{h}{2}\left(1+\tfrac{1}{\sqrt{3}}\right)\right)$$

$$\Rightarrow \quad A = \frac{h}{2}\sqrt{\tfrac{h}{2}\left(1-\tfrac{1}{\sqrt{3}}\right)} \qquad B = \frac{h}{2}\sqrt{\tfrac{h}{2}\left(1+\tfrac{1}{\sqrt{3}}\right)}$$

$$a = \frac{1}{2}\left(1 - \tfrac{1}{\sqrt{3}}\right) \qquad b = \frac{1}{2}\left(1 + \tfrac{1}{\sqrt{3}}\right)$$

Possible deficiencies: $\sqrt{x}\, f(x)$ may be hard to interpolate
or $f(x)$ isn't defined at these points

b) • Choose inner-product:

weight func

$$\langle p, q \rangle = \int_0^h p(x)\, q(x) \sqrt{x}\, dx$$

• Find orthogonal polynomials: (Need quadratic)

Basis: $\{1, x, x^2\}$

$$u_1 = 1$$

$$u_2 = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1 = x - \frac{\int_0^h x^{3/2}\, dx}{\int_0^h x^{1/2}\, dx} = x - \frac{\frac{2}{5} h^{5/2}}{\frac{2}{3} h^{3/2}} = x - \frac{3}{5} h$$

$$u_3 = x^2 - \frac{\langle x^2, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} 1$$

• Find roots of u_3, x_0 and x_1.

$$\Rightarrow a = \frac{x_0}{h} \quad \text{and} \quad b = \frac{x_1}{h}$$

• Find weights

$$f(x) = 1: \quad \int_0^h x^{1/2}\, dx = \frac{2}{3} h^{3/2} = A f(x_0) + B f(x_1) = A + B$$

$$f(x) = x: \quad \int_0^h x^{3/2}\, dx = \frac{2}{5} h^{5/2} = A f(x_0) + B(x_1) = A x_1 + B x_2$$

$$\begin{cases} A + B = \frac{2}{3} h^{3/2} \\ A x_1 + B x_2 = \frac{2}{5} h^{5/2} \end{cases}$$

Solve this system for A and B

Problem 6

Jan 2016
+ Jan 2017

$$\int_0^h \sqrt{x}\, f(x)dx = A f(ah) + B f(bh) + Ch^p$$

a) $$\int_0^h \sqrt{x}\, f(x)\,dx = \int_{-1}^{1} \underbrace{\sqrt{\tfrac{h}{2}(y+1)}\, f\left(\tfrac{h}{2}(y+1)\right)\tfrac{2}{h}}_{F(y)}\, dy \approx F\left(-\tfrac{1}{\sqrt{3}}\right) + F\left(\tfrac{1}{\sqrt{3}}\right)$$

$y = \frac{2}{h}x - 1$

$x = \frac{h}{2}(y+1)$

$$I(f) \approx \sqrt{\tfrac{h}{2}(1-\tfrac{1}{\sqrt{3}})}\; f\left(\tfrac{h}{2}(1-\tfrac{1}{\sqrt{3}})\right) + \sqrt{\tfrac{h}{2}(1+\tfrac{1}{\sqrt{3}})}\; f\left(\tfrac{h}{2}(1+\tfrac{1}{\sqrt{3}})\right)$$

$$\Rightarrow A = \sqrt{\tfrac{h}{2}(1-\tfrac{1}{\sqrt{3}})} \qquad B = \sqrt{\tfrac{h}{2}(1+\tfrac{1}{\sqrt{3}})}$$

$$a = \frac{1-\frac{1}{\sqrt{3}}}{2} \qquad b = \frac{1+\frac{1}{\sqrt{3}}}{2}$$

Possible defficiencies: If $\sqrt{x}\, f(x)$ is hard to interpolate.

b) • Choose inner product: $\langle p, q\rangle = \int_0^h p(x)q(x)\sqrt{x}\, dx$ ← weighting function

• Find orthogonal polynomials: (Need a quadratic)
basis $\{1, x, x^2\}$

$u_1 = 1$

$$u_2 = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle}\cdot 1 = x - \frac{3h}{5}$$

$$u_3 = x^2 - \frac{\langle x^2, x\rangle}{\langle x, x\rangle}\cdot u_2 - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle}\cdot 1 = x^2 - \frac{10h}{9}x + \frac{5h^2}{21}$$

• Find roots of u_3:

$$x_0 = \frac{35h - 2h\sqrt{70}}{63}$$

$$x_1 = \frac{35 + 2h\sqrt{70}}{63}$$

• Find weights:

$f(x) = 1$: $\int_0^h \sqrt{x}\, dx = \frac{2}{3}h^{3/2} = A + B$

$f(x) = x$: $\int_0^h \sqrt{x}\cdot x\, dx = \frac{2}{5}h^{5/2} = A x_0 + B x_1$

$$\Rightarrow \boxed{A = \frac{h^{3/2}}{150}(50 - \sqrt{70})} \quad \boxed{B = \frac{h^{3/2}}{150}(50 + \sqrt{70})}$$

Note: Don't actually find A and B on comps.

Hermite interpolating polynomial

$$\int_0^h f(x)\sqrt{x}\,dx - Q(f,h) = \int_0^h f(x)\sqrt{x}\,dx - \int_0^h p(x)\sqrt{x}\,dx$$

$$= \int_0^h \left(f(x) - p(x)\right)\sqrt{x}\,dx$$

$$= \int_0^h \frac{f'''(\xi)}{6} \prod_{i=0}^{1} (x - x_i)^2 \sqrt{x}\,dx$$

$$= \frac{f''(\xi)}{6} \int_0^h \prod_{i=0}^{1} (x - x_i)^2 \sqrt{x}\,dx = O(h^{11/2})$$

C = The coeff of $h^{11/2}$