METHODS OF APPLIED MATHEMATICS COMPREHENSIVE EXAMINATION AUGUST 2014

Work on as many of the following problems as possible. Turn in *all* your work.

(1) Consider a projectile fired vertically from the surface of the Earth. Assume there is an inverse quadratic, attractive gravitational force field ($F = -\frac{GM_e m}{R^2}$, where R distance between the projectile and the Earth's center, M_e, m are the respective Earth and projectile mass, and G is the universal gravitational constant.) Assume the projectile experiences a drag force proportional to the cube of its velocity.

(a) Derive an equation for the projectile's dynamics assuming that the Earth is fixed, decouple the dynamics.

(b) Measure the projectile height from the Earth surface by introducing R_e as the radius of the Earth. Then, identify standard kinematics by letting $g = \frac{GM_e}{R_e^2} \approx 9.8 m/s^2$. Identify characteristic length and time scales in terms of g and the initial velocity, v_0. Non-dimensionalize the problem, and identify the non-dimensional parameter and simplified non-linear ODE governing the evolution.

(c) Assuming that the projectile speed is sufficiently small, write down a two term perturbation expansion for the time to return to the Earth surface.

(2) Consider the following integral with real parameters $a > b > 0$,

$$I(a,b) \equiv \int_0^{2\pi} \log(a + b\cos\theta)d\theta$$

By complexifying the integrand by $z = e^{i\theta}$ turn this into a contour integration on the unit circle and evaluate the integral by paying close attention to the branch structure of the integrand.

(3) Consider the rapidly varying diffusivity:

$$K(x,y;\epsilon) = A + F(x/\epsilon) + G(y/\epsilon^2)$$

where A is chosen to guarantee K is positive, and ϵ is a small constant. By applying iterated homogenization, average the following diffusion equation to derive a leading order effective equation governing the evolution as $\epsilon \to 0$ over the (x,y)-plane, assuming the functions $F(x)$ and $G(y)$ are mean zero, periodic, and share the same period:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K(x,y;\epsilon)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K(x,y;\epsilon)\frac{\partial u}{\partial y}\right)$$
$$u(x,y,0) = u_0(x,y)\,.$$

(4) Find two term asymptotic expansions as $\epsilon \to 0$ for all roots of the equation:

$$\epsilon z^3 + z^2 + 2z + 1 = \epsilon$$

(5) Consider the following initial value problem for the equation

$$u_t + xu_x = \epsilon u u_x\,, \qquad u(x,0) = u_0(x)\,,$$

where ϵ is a fixed nonzero small real parameter.

(a) Solve with the method of characteristics for $\epsilon = 0$ and all times $t \geq 0$ on the real line x assuming the initial condition $u_0(x)$ is a "nice" function decaying as $|x| \to \infty$.

(b) Set up a regular perturbation expansion for $u(\cdot,\cdot,\epsilon)$ and find the first correction to the leading order solution found above.

(6) Consider the eigenvalue problem on the real line $x \in \mathbb{R}$.

$$\epsilon y'' - (U(x) + \lambda)y = 0.$$

with the (square well) potential $U(x) = 1$ for $|x| > 1$ and $U(x) = -1$ for $|x| < 1$

Need $\lambda \in (-1,1)$ to get bdd soln

(a) Solve for eigenvalues and eigenfunctions exactly.

(b) Study their asymptotic limit as $\epsilon \to 0$.

(c) Compare with the results obtained by a WKBJ approach in this limit.

(7) Consider the 6th-order ordinary differential equation on the real line $x \in \mathbb{R}$,

$$\frac{d^6 y}{d\,x^6} = xy\,.$$

(a) By using a contour integral representation in the appropriate complex Laplace-image plane, discuss whether solutions decaying as $|x| \to \infty$ exist.

(b) Find the leading order asymptotic expansion of one of these solutions as $x \to \infty$ by appropriate deformations of the contour integral representation.

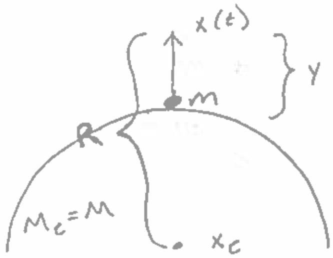
(8) (a) Explain the difference between pointwise convergence and asymptotic convergence. Illustrate with the particular example of power series.

(b) Consider the map that associates to a function $f(z)$ its asymptotic expansion as $z \to z_0$, where z_0 is a point of the complex plane, including infinity. Is this mapping one-to-one (injective), onto (surjective), one-to-one and onto (bijective), or none of these properties applies? Explain.

(c) Is the asymptotic relation distributive? Discuss.

Problem 1

Aug 2014



$$m\frac{d^2x}{dt^2} = -\frac{GMm}{(x-x_e)^2} + \beta\left(\frac{dx}{dt}\right)^3$$

$$M\frac{d^2x_e}{dt^2} = \frac{GMm}{(x-x_e)^2} = \text{const by assumption}$$

a) $$m\frac{d^2x}{dt^2} = -\frac{GMm}{(x-x_e)^2} - \beta\left(\frac{dx}{dt}\right)^3$$

$$\Rightarrow \frac{d^2x}{dt} = -\frac{GM}{R^2} - \frac{\beta}{m}\left(\frac{dx}{dt}\right)^3$$

where $x(t)\big|_{t=0} = R_e$ = radius of Earth

and $\frac{dx}{dt}\big|_{t=0} = v_0$

b) Change of variables:

Let $\begin{cases} y = x - R_e \\ x_e = 0 \end{cases} \Rightarrow \begin{cases} x = y + R_e \\ x_e = 0 \end{cases}$

$$\Rightarrow \frac{d^2y}{dt^2} = -\frac{GM}{(y+R_e)^2} + \frac{\beta}{m}\left(\frac{dy}{dt}\right)^3$$

$$\boxed{\frac{d^2y}{dt^2} = -\frac{GM}{R_e^2}\cdot\frac{1}{\left(1+\frac{y}{R_e}\right)^2} - \frac{\beta}{m}\left(\frac{dy}{dt}\right)^3} \qquad (1)$$

where $g = \frac{GM}{R_e^2}$. Since $y \ll R_e$, we can say:

<u>Kinematics</u>

$y'' = -g$

$y' = -gt + b \qquad y'(0) = b = v_0$

$y = -\frac{1}{2}gt^2 + v_0 t + c \qquad y(0) = c = 0$

$y'(t_a) = -g t_a + v_0 = 0 \Rightarrow t_a = \frac{v_0}{g}$

$$\Rightarrow y_{max} = y(t_a) = -\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\left(\frac{v_0}{g}\right) = \frac{v_0^2}{g} - \frac{1}{2}\frac{v_0^2}{g} = \frac{v_0^2}{2g}$$

Non-Dimensionalization

Choose $L = \frac{V_0^2}{g}$ and $T = \frac{V_0}{g}$. Let $y = zL$ and $t = \tau T$

then :

$$\frac{dy}{dt} = \frac{d(zL)}{dt} = L \frac{dz}{d\tau} \frac{d\tau}{dt} = \frac{L}{T} \frac{dz}{d\tau}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{L}{T}\frac{dz}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{L}{T}\frac{dz}{d\tau}\right)\frac{d\tau}{dt} = \frac{L}{T^2}\frac{d^2z}{d\tau^2}$$

So (1) becomes:

$$\underbrace{\frac{L}{T^2}}_{g} \frac{d^2 z}{d\tau^2} = -g \cdot \frac{1}{\left(1 + \frac{Lz}{R_E}\right)^2} - \frac{\beta}{m}\left(\frac{L}{T}\frac{dz}{d\tau}\right)^3$$

$$\Rightarrow z_{\tau\tau} = -\frac{1}{(1+\varepsilon z)^2} - \frac{\beta}{m}\frac{L^2}{T}\left(\frac{dz}{d\tau}\right)^3$$

$$z_{\tau\tau} = -\frac{1}{(1+\varepsilon z)^2} - \alpha\varepsilon\left(\frac{dz}{d\tau}\right)^3$$

where: $\varepsilon = \frac{L}{R_E} = \frac{V_0^2}{gR_E}$

and $\alpha = \frac{\beta R_E}{m}\frac{L}{T} = \frac{\beta V_0 R_E}{m}$

Note that $z\big|_{\tau=0} = 0$ and

$$V_0 = \frac{dy}{dt}\Big|_{t=0} = \frac{dy}{d\tau}\frac{d\tau}{dt}\Big|_{\tau=0} = \frac{1}{T}\frac{d(zL)}{d\tau}\Big|_{\tau=0} = \frac{L}{T}\frac{dz}{d\tau}\Big|_{\tau=0} = V_0\frac{dz}{d\tau}\Big|_{\tau=0}$$

$$\Rightarrow \begin{cases} z_{tt} = -\frac{1}{(1+\varepsilon z)^2} - \alpha\varepsilon\left(\frac{dz}{dt}\right)^3 \\ z(0) = 0 \\ z'(0) = 1 \end{cases} \qquad (2)$$

c) For sufficiently small v_0, we get $\varepsilon \ll 1$. So consider the following:

$$z = z_0 + \varepsilon z_1 + \varepsilon^2 z_2 + \dots$$

Use this in (2) to get a hierarchy of equations:

$$(z_0'' + \varepsilon z_1'' + \dots) = - \frac{1}{(1+\varepsilon(z_0 + \varepsilon z_1 + \dots))^2} - \alpha \varepsilon (z_0' + \varepsilon z_1' + \dots)^3$$

$$= -1 + 2\varepsilon(z_0 + \varepsilon z_1 + \dots) + \mathcal{O}(\varepsilon^2) - \alpha \varepsilon (z_0' + \varepsilon z_1' + \dots)^3$$

$\mathcal{O}(1)$: $z_0'' = -1$

$$z_0' = -\tau + a \qquad z_0'(0) = a = 1$$

$$z_0 = -\tfrac{1}{2}\tau^2 + \tau + b \qquad z_0(0) = b = 0$$

$\mathcal{O}(\varepsilon)$: $z_1'' = 2z_0 - \alpha\varepsilon z_0' = -t^2 + 2t + \alpha(-t+1)^3$

$$z_1' = -\tfrac{1}{3}\tau^3 + \tau^2 - \tfrac{\alpha}{4}(-\tau+1)^4 + c \qquad z_1'(0) = c - \tfrac{\alpha}{4} = 0$$

$$z_1 = -\tfrac{1}{12}\tau^4 + \tfrac{1}{3}\tau^3 - \tfrac{\alpha}{20}(-\tau+1)^5 - \tfrac{\alpha}{4}\tau + d \qquad z_1(0) = -\tfrac{\alpha}{20} + d = 0$$

$$z_1 = -\tfrac{1}{12}\tau^4 + \tfrac{1}{3}\tau^3 - \tfrac{\alpha}{20}(-\tau+1)^5 - \tfrac{\alpha}{4}\tau + \tfrac{\alpha}{20}$$

$$\Rightarrow z \approx \left(-\tfrac{1}{2}\tau^2 + \tau\right) + \varepsilon\left(-\tfrac{\alpha}{20}(-\tau+1)^5 - \tfrac{1}{12}\tau^4 + \tfrac{1}{3}\tau^3 - \tfrac{\alpha}{4}\tau + \tfrac{\alpha}{20}\right) \qquad (3)$$

The return time fulfills $z(\tau_r) = 0$.

Ansatz: $\tau_r = \tau_0 + \varepsilon \tau_1 + \ldots$

Plug this into (3) = 0 and get a hierarchy of eqns:

$$\mathcal{O}(1): \quad -\frac{1}{2}\tau_0^2 + \tau_0 = 0$$

$$\tau_0\left(-\frac{1}{2}\tau_0 + 1\right) = 0$$

$$\tau_0 = 0 \qquad \tau_0 = 2$$

$\uparrow$ start time $\qquad$ $\uparrow$ return time

$$\mathcal{O}(\varepsilon): \quad -\tau_0\tau_1 + \tau_1 = \frac{\alpha}{20}(-\tau_0+1)^5 + \frac{1}{12}\tau_0^4 - \frac{1}{3}\tau_0^3 + \frac{\alpha}{4}\tau_0 + \frac{\alpha}{20}$$

$$= -\cancel{\frac{\alpha}{20}} + \frac{2}{3} - \frac{8}{3} + \frac{\alpha}{2} + \cancel{\frac{\alpha}{20}} = \frac{\alpha}{2} - 3$$

$$\tau_1 = 3 - \frac{\alpha}{2}$$

$$\therefore \boxed{\tau_{rc} = 2 + \varepsilon\left(3 - \frac{\alpha}{2}\right)}$$

Problem 2

$$I(a,b) = \int_0^{2\pi} \log(a + b\cos\theta)\, d\theta$$

Let $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta$

$$\Rightarrow I = \int_C \log\left(a + \frac{b}{2}\left(z + \frac{1}{z}\right)\right)\left(\frac{-i}{z}\right) dz \qquad \text{where } C = \text{unit circle}$$

$$= -i \int_C \frac{\log\left(a + \frac{b}{2}\left(z + \frac{1}{z}\right)\right)}{z}\, dz$$

$$= -i \int_C \frac{\log\left(\frac{b}{2}z^2 + az + \frac{b}{2}\right) - \log z}{z}\, dz$$

$a > b$ so is real

$$z = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

$$= -i\left(\int_C \frac{\overset{z_1=0}{\log\left(z - \left(\frac{-a + \sqrt{a^2 - b^2}}{b}\right)\right)}}{z} dz + \int_C \frac{\overset{z_0=0}{\log\left(z - \left(\frac{-a - \sqrt{a^2 - b^2}}{b}\right)\right)}}{z} dz - \int_C \frac{\overset{z_1=0}{\log z}}{z}\, dz\right)$$

$$\left|\frac{-a + \sqrt{a^2 - b^2}}{b}\right| \overset{?}{<} 1$$

$$|-a + \sqrt{a^2-b^2}| < b$$

$$a - \sqrt{a^2 - b^2} < b$$

$$a - b < \sqrt{a^2 - b^2}$$

$$a^2 - 2ab + b^2 < a^2 - b^2$$

$$2b^2 < 2ab$$

$$b < a \checkmark$$

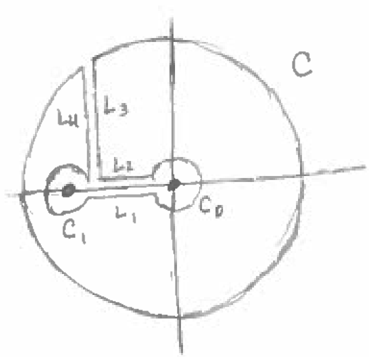
$$\left|\frac{-a - \sqrt{a^2 - b^2}}{b}\right| \overset{?}{>} 1$$

$$a + \sqrt{a^2 - b^2} > b$$

$$a - b > -\sqrt{a^2 - b^2} \checkmark$$

$$= -i \int_C \frac{\log\left(1 - \frac{-a + \sqrt{a^2 - b^2}}{bz}\right)}{z} dz + i \underbrace{\int_C \frac{\log\left(z - \frac{-a - \sqrt{a^2 - b^2}}{b}\right)}{z}}_{\operatorname{Res}\left(\log\left(z - \frac{-a - \sqrt{a^2-b^2}}{b}\right),\, 0\right)}$$

$$= 2\pi i \log\left(\frac{a + \sqrt{a^2 - b^2}}{b}\right)$$



Problem 3

$$K(x,y,\varepsilon) = A + F\left(\tfrac{x}{\varepsilon}\right) + G\left(\tfrac{y}{\varepsilon^2}\right)$$

$$u_t = \partial_x(K\partial_x u) + \partial_y(K\partial_y u)$$

$$u(x,y,0) = u_0(x,y)$$

Let $\begin{cases} w = \frac{y}{\varepsilon^2} \\ z = \frac{x}{\varepsilon} \end{cases} \Rightarrow \begin{cases} \partial_y \longmapsto \partial_y + \frac{1}{\varepsilon^2}\partial_w \\ \partial_x \longmapsto \partial_x + \frac{1}{\varepsilon}\partial_z \end{cases}$

$$\Rightarrow u_t = \left(\partial_x + \tfrac{1}{\varepsilon}\partial_z\right)\left(K\left(\partial_x + \tfrac{1}{\varepsilon}\partial_z\right)u\right) + \left(\partial_y + \tfrac{1}{\varepsilon^2}\partial_w\right)\left(K\left(\partial_y + \tfrac{1}{\varepsilon^2}\partial_w\right)u\right)$$

Ansatz: $u(x,y,w,z,t) = \bar{u}(x,y,w,z,t) + \varepsilon u_1(x,y,z,w,t) + \dots$

$\mathcal{O}\left(\frac{1}{\varepsilon^4}\right)$: $\partial_w(K\partial_w\bar{u}) = 0 \Rightarrow \bar{u}(x,y,z,t)$

$\mathcal{O}\left(\frac{1}{\varepsilon^3}\right)$: $\partial_w(K\partial_w u_1) = 0 \Rightarrow u_1(x,y,z,t)$

$\mathcal{O}(1/\varepsilon^2)$: $\partial_w(K\partial_w u_2) + \cancelto{0}{\partial_y(K\partial_w\bar{u})} + \partial_w(K\partial_y\bar{u}) + \partial_z(K\partial_z\bar{u}) = 0$

$$\langle\partial_w(K\partial_w u_2 + K\partial_y\bar{u})\rangle_w + \underbrace{\cancelto{0}{\langle\partial_z K\partial_z\bar{u}\rangle_w}}_{FA} = 0$$

$$K\partial_w u_2 + K\partial_y\bar{u} = \langle K\rangle_w^h\,\partial_y\bar{u} \qquad \Rightarrow \quad \bar{u}(x,y,t)$$

$\mathcal{O}(1/\varepsilon)$: $\partial_w(K\partial_w u_3) + \cancelto{0}{\partial_y(K\partial_w u_1)} + \partial_w(K\partial_y u_1) + \partial_z(K\partial_z u_1)$

$$+ \partial_z(K\partial_x\bar{u}) + \cancelto{0}{\partial_x(K\partial_z\bar{u})} = 0$$

$$\cancelto{0}{\langle\partial_w(K\partial_w u_3 + K\partial_y u_1)\rangle_w} + \langle\partial_z(K\partial_z u_1 + K\partial_x\bar{u})\rangle_w = 0$$

$$\langle K\rangle_w(\partial_z u_1 + \partial_x\bar{u}) = \langle\langle K\rangle_w\rangle_z^h\,\partial_x\bar{u}$$

$$O(1):\ \partial_w(K\partial_w u_4)+\partial_y(K\partial_w u_2)+\partial_w(K\partial_y u_2)+\partial_z(K\partial_z u_2)+\partial_z(K\partial_x u_1)$$
$$+\partial_x(K\partial_z u_1)+\partial_x(K\partial_x\bar{u})+\partial_y(K\partial_y\bar{u})=\bar{u}_t$$

$$\cancel{\langle\partial_w(K(\partial_w u_4+\partial_y u_2))\rangle_w}^{\,0}+\langle\partial_z(K(\partial_z u_2+\partial_x u_1))\rangle_w$$
$$+\langle\partial_y(K(\partial_w u_2+\partial_y\bar{u}))\rangle_w+\langle\partial_x(K(\partial_z u_1+\partial_x\bar{u}))\rangle_w=\langle\bar{u}_t\rangle_w=\bar{u}_t$$

$$\cancel{\langle\partial_z(\langle K\rangle_w(\partial_z u_2+\partial_x u_1))\rangle_z}^{\,0}+\langle\partial_y\langle K\rangle_w^h\partial_y\bar{u}\rangle_z+\langle\partial_x\underbrace{(\langle K\rangle_w(\partial_z u_1+\partial_x\bar{u}))}_{\langle\langle K\rangle_w\rangle_z^h\partial_x\bar{u}}\rangle_z=\langle\bar{u}_t$$

$$\therefore\ \boxed{\bar{u}_t=\langle\langle K\rangle_w\rangle_z^h\bar{u}_{xx}+\langle\langle K\rangle_w^h\rangle_z\bar{u}_{yy}}$$

Problem 4

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$$\varepsilon z^3 + z^2 + 2z + 1 = \varepsilon \qquad (1)$$

$$\Rightarrow \quad \varepsilon z^3 + (z+1)^2 = \varepsilon \qquad (2)$$

Let $\varepsilon = 0 \quad \Rightarrow \quad (z+1)^2 = 0 \quad \Rightarrow \quad z_{1,2} = -1$

$\Rightarrow \quad z_{1,2} \sim -1 + a_1 \varepsilon^{1/2}$ as $\varepsilon \to 0$

Plug into (2):

$$\varepsilon(-1 + a_1\varepsilon^{1/2})^3 + (-1 + a_1\varepsilon^{1/2} + 1)^2 = \varepsilon$$

$$\varepsilon(-1 + a_1\varepsilon^{1/2})^3 + (a_1\varepsilon^{1/2})^2 = \varepsilon$$

$$-\varepsilon + 3a_1\varepsilon^{\frac{1}{2}+1} - 3a_1^2\varepsilon^{2\cdot\frac{1}{2}+1} + a_1\varepsilon^{3\cdot\frac{1}{2}+1} + a_1^2\varepsilon^{2\cdot\frac{1}{2}} = \varepsilon$$

Simplify:

$$3a_1\varepsilon^{3/2} - 3a_1\varepsilon^2 + a_1\varepsilon^{5/2} + a_1^2\varepsilon = 2\varepsilon$$

$\Theta(\varepsilon)$: $\quad a_1^2 = 2 \quad \Rightarrow \quad a_1 = \pm\sqrt{2}$

$$\therefore \boxed{z_{1,2} \sim -1 \pm \sqrt{2}\,\varepsilon^{1/2}}$$

$z_3 \sim -\frac{1}{\varepsilon} + b$ as $\varepsilon \to 0$

Plug into (1)

$$\varepsilon\left(-\tfrac{1}{\varepsilon} + b\right)^3 + \left(-\tfrac{1}{\varepsilon} + b\right)^2 + 2\left(-\tfrac{1}{\varepsilon} + b\right) + 1 = \varepsilon$$

$$\varepsilon\left(-\tfrac{1}{\varepsilon^3} + \tfrac{3}{\varepsilon^2}b - \tfrac{3}{\varepsilon}b^2 + b^3\right) + \tfrac{1}{\varepsilon^2} - \tfrac{2}{\varepsilon}b + b^2 - \tfrac{2}{\varepsilon} + 2b + 1 = \varepsilon$$

$\Theta\left(\frac{1}{\varepsilon}\right)$: $\quad 3b - 2b - 2 = 0$

$$b = 2$$

$$\Rightarrow \boxed{z_3 = -\tfrac{1}{\varepsilon} + 2}$$

Problem 5

$$\begin{cases} u_t + x u_x = \varepsilon u u_x \\ u(x,0) = u_0(x) \end{cases} \qquad (1)$$

a) Let $\varepsilon = 0 \Rightarrow u_t + x u_x = 0$ (2)

Define $\begin{cases} z(s) = u(X(s), T(s)) \\ X(0) = x \\ T(0) = t \end{cases}$

From the coefficients in (2) we know:

$$\begin{cases} \frac{dX}{ds} = X(s) \\ \frac{dT}{ds} = 1 \end{cases} \Rightarrow \begin{cases} X(s) = a e^{s} \\ T(s) = s + b \end{cases} \Rightarrow \begin{cases} X(s) = x e^{s} \\ T(s) = s + t \end{cases}$$

by I.C.

By choice of z: $\frac{dz}{ds} = 0$

$\Rightarrow z(s) = z(0)$

$z(s) = u(X(s), T(s)) = u(xe^{s}, s+t)$

$\Rightarrow z(0) = u(x,t)$

Now we choose an appropriate s: $s = -t$

$\Rightarrow z(-t) = u(xe^{-t}, 0) = u_0(xe^{-t})$

$z(0) = z(-t)$

$\therefore \boxed{u(x,t) = u_0(xe^{-t})}$

b) Ansatz: $u(x,t) = \bar{u}(x,t) + \varepsilon u_1(x,t) + \ldots$

where $\bar{u}(x,0) = u_0(x)$

$u_i(x,0) = 0 \qquad \forall\, i = 1,2,3,\ldots$

$\mathcal{O}(1)$: $\bar{u}_t + x\bar{u}_x = 0$

$\Rightarrow \bar{u} = u_0(xe^{-t})$ from part a

$\mathcal{O}(\varepsilon)$: $u_{1_t} + xu_{1_x} = \bar{u}\bar{u}_x = f(x,t)$

Define $\begin{cases} z(s) = u_1(X(s), T(s)) \\ X'(s) = X, \quad T'(s) = 1 \end{cases} \Rightarrow X(s) = xe^{s} \qquad T(s) = s+t$

$\frac{dz}{ds} = \bar{u}\bar{u}_x = g(X(s), T(s))$

$z(s) = z(0) + \int_0^s g(X(v), T(v))\, dv$

choose $s = -t$

$z(-t) = z(0) + \int_0^{-t} g(X(v), T(v))\, dv = u_1(xe^{-t}, 0) = 0$

where $z(0) = u_1(x,t)$

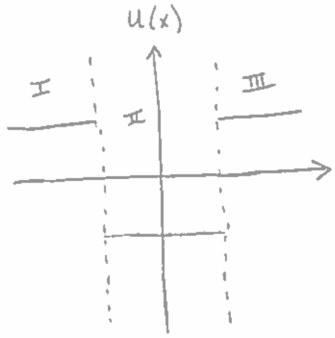
$\therefore \boxed{u_1(x,t) = -\int_0^{-t} g(xe^{v}, v+t)\, dv}$

Problem 6

Aug 2014

$$\varepsilon y'' - (U(x)+\lambda) y = 0 \qquad (1)$$

$$\Rightarrow y'' = \left(\frac{U(x)+\lambda}{\varepsilon}\right) y \qquad (2)$$



$$U(x) = \begin{cases} 1 & |x|>1 \\ -1 & |x|<1 \end{cases}$$

a) **Case 1:** $\lambda > 1$

Region I: $U(x)=1$

$$y'' = \left(\frac{\lambda+1}{\varepsilon}\right) y = 0$$

$$y_1 = A e^{\sqrt{\frac{\lambda+1}{\varepsilon}}\,x} + B e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}\,x}$$

Region II: $U(x) = -1$

$$y'' - \left(\frac{\lambda-1}{\varepsilon}\right) y = 0$$

$$y_2 = C e^{\sqrt{\frac{\lambda-1}{\varepsilon}}\,x} + D e^{-\sqrt{\frac{\lambda-1}{\varepsilon}}\,x}$$

Patch: $y_1(-1) = y_2(-1)$

$$A e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}} + B e^{\sqrt{\frac{\lambda+1}{\varepsilon}}} = C e^{-\sqrt{\frac{\lambda-1}{\varepsilon}}} + D e^{\sqrt{\frac{\lambda-1}{\varepsilon}}}$$

$$y_1'(-1) = y_2'(-1)$$

$$A\sqrt{\frac{\lambda+1}{\varepsilon}}\, e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}} - B\sqrt{\frac{\lambda+1}{\varepsilon}}\, e^{\sqrt{\frac{\lambda+1}{\varepsilon}}} = C\sqrt{\frac{\lambda-1}{\varepsilon}}\, e^{-\sqrt{\frac{\lambda-1}{\varepsilon}}} - D\sqrt{\frac{\lambda-1}{\varepsilon}}\, e^{\sqrt{\frac{\lambda-1}{\varepsilon}}}$$

<u>Region III</u>: $U(x) = 1$

$$y_3 = E e^{\sqrt{\frac{\lambda+1}{\varepsilon}}\,x} + F e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}\,x}$$

Patch: $y_2(1) = y_3(1)$

$$C e^{\sqrt{\frac{\lambda-1}{\varepsilon}}} + D e^{-\sqrt{\frac{\lambda-1}{\varepsilon}}} = E e^{\sqrt{\frac{\lambda+1}{\varepsilon}}} + F e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}}$$

$$y_2'(1) = y_2'(1)$$

$$C\sqrt{\tfrac{\lambda-1}{\varepsilon}}\, e^{\sqrt{\frac{\lambda-1}{\varepsilon}}} - D\sqrt{\tfrac{\lambda-1}{\varepsilon}}\, e^{-\sqrt{\frac{\lambda-1}{\varepsilon}}} = E\sqrt{\tfrac{\lambda+1}{\varepsilon}}\, e^{\sqrt{\frac{\lambda+1}{\varepsilon}}} - F\sqrt{\tfrac{\lambda+1}{\varepsilon}}\, e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}}$$

$$\begin{bmatrix} B & A & -D & -C \\ -B\sqrt{\frac{\lambda+1}{\varepsilon}} & A\sqrt{\frac{\lambda+1}{\varepsilon}} & D\sqrt{\frac{\lambda-1}{\varepsilon}} & -C\sqrt{\frac{\lambda-1}{\varepsilon}} \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} e^{\sqrt{\lambda+1/\varepsilon}} \\ e^{-\sqrt{\frac{\lambda+1}{\varepsilon}}} \\ e^{\sqrt{\lambda-1/\varepsilon}} \\ e^{-\sqrt{\frac{\lambda-1}{\varepsilon}}} \end{bmatrix} = \vec{0}$$

Problem 7

$$\frac{d^6 y}{dx^6} = xy \qquad (1)$$

a) For some path γ:

$$y(x) = \int_\gamma e^{sx}\hat{y}(s)\,ds \qquad (2)$$

$$y' = \int_\gamma e^{sx} s\,\hat{y}(s)\,ds$$

$$\vdots$$

$$y^{(6)} = \int_\gamma e^{sx} s^6 \hat{y}(s)\,ds$$

Plug into (1):

$$\int_\gamma e^{sx} s^6 \hat{y}(s)\,ds = x\int_\gamma e^{sx}\hat{y}(s) = \int_\gamma x e^{sx}\hat{y}(s)\,ds$$

$$\int_\gamma e^{sx} s^6 \hat{y}(s)\,ds = -\int_\gamma e^{sx}\hat{y}'(s)\,ds$$ by Int. by Parts and choosin γ st the endpoints cancel

$$\int_\gamma e^{sx}\left(s^6\hat{y}(s) + \hat{y}'(s)\right) = 0$$

$$\hat{y}'(s) + s^6\hat{y}(s) = 0$$

$$\rho(s) = e^{\int s^6\,ds} = e^{\frac{1}{7}s^7}$$

$$\Rightarrow \quad e^{\frac{1}{7}s^7}\hat{y}'(s) + s^6 e^{\frac{1}{7}s^7}\hat{y}(s) = 0$$

$$\frac{d}{ds}\left(e^{\frac{1}{7}s^7}\hat{y}(s)\right) = 0$$

$$\hat{y}(s) = a\,e^{-\frac{1}{7}s^7}$$

Plug into (2):

$$y(x) = a\int_\gamma e^{sx - \frac{1}{7}s^7}\,ds$$

$$y(x) = a\int_\gamma e^{sx - \frac{1}{7}s^7}\,ds$$

For our soln to decay, we need $y \to 0$ as $|x| \to \infty$ which occurs if $Re(\frac{s^7}{7}) > 0$.

$$s = x^\alpha u$$

$$\Rightarrow \begin{cases} xs = x^{\alpha+1}u \\ s^7 = x^{7\alpha}u^7 \\ ds = x^\alpha du \end{cases} \Rightarrow x^{\alpha+1} = x^{7\alpha} \Rightarrow \alpha = \frac{1}{6}$$

$$y(x) = a\int_\gamma e^{x^{7/6}(u - \frac{u^7}{7})}\, x^{1/6}\,du = a x^{1/6}\int_\gamma e^{x^{7/6}(u-\frac{u^7}{7})}\,du$$

$$h(u) = u - \frac{u^7}{7}$$

$$h'(u) = 1 - u^6 \qquad 1 - u_0^6 = 0 \Rightarrow u_0 = e^{2\pi i k/6} = e^{\pi i k/3} \quad k = 0, \dots, 5$$

$$h''(u) = -6u^5$$

$$h''(e^{\pi i k/3}) = -6u^{5\pi i k/3} = 6e^{i\pi}e^{5\pi i k/3}$$

$$= 6e^{i\pi(1 + 5k/3)} \neq 0$$

$$\Rightarrow \alpha = \pi + \frac{5\pi k}{3} \qquad k = 0, \dots, 5$$

$$\Rightarrow \theta = -\frac{\alpha}{n} + (2m+1)\frac{\pi}{n} \qquad n = 2 \Rightarrow m = 1 \qquad \text{(angle of S.D.)}$$

$$= -\frac{\pi}{2} + \frac{5\pi k}{6} + \frac{3\pi}{2} = \pi + \frac{5\pi k}{6}$$

Problem 7

$$\frac{d^6 y}{dx^6} = xy \qquad (1)$$

$$y = f(x) = \int_C F(s)\, e^{sx}\, ds$$

$$f^{(6)}(x) = \int_C F(s)\, s^6 e^{sx}\, ds$$

$$x f(x) = \int_C F(s)\, x e^{sx}\, ds = \int_C F(s) \frac{d}{ds} e^{sx}\, ds$$

$$= e^{sx} F(s) \Big|_{C^-}^{C^+} - \int_C F'(s)\, e^{sx}\, ds$$

$$\Rightarrow \int_C \left(s^6 F(s) + F'(s) \right) e^{sx}\, ds = 0$$

Cond on choosing C:

i) convergent integrals

ii) $e^{st} F(s) \Big|_C = 0$

$$-s^6 F(s) = F'(s)$$

$$F(s) = F(0)\, e^{-s^7/7}$$

$$y = f(x) = A \int e^{sx - r^7/7}\, ds \qquad \text{where} \qquad s = re^{i\theta}$$

The behavior of

$$e^{re^{i\theta} x - \frac{r^7 e^{7i\theta}}{7}} \quad \text{as} \quad r \to \infty \qquad \theta \in [0, 2\pi]$$

is determined by

$$\left| e^{-\frac{r^7}{7} e^{7i\theta}} \right| = e^{-r^7/7 \cos 7\theta} < \infty \quad \text{if} \quad \cos 7\theta > 0$$

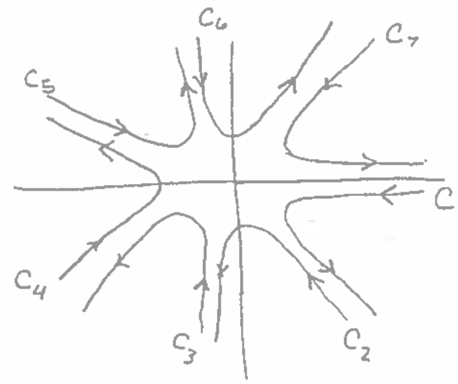
$$-\frac{\pi}{2} + 2k\pi < 7\theta < \frac{\pi}{2} + 2k\pi$$

$k = 0$: $-\frac{\pi}{14} < \theta < \frac{\pi}{14}$

$k = 1$: $\frac{3\pi}{14} < \theta < \frac{5\pi}{14}$

$\vdots$

$k = 6$: $\frac{23\pi}{14} < \theta < \frac{25\pi}{14}$



$$\begin{cases} C_1: \quad \infty \longrightarrow \infty e^{\frac{24i\pi}{14}} \\ C_2: \\ \vdots \\ C_7 \quad \infty e^{4\pi i/14} \longrightarrow \infty \end{cases}$$

$$s = x^{\alpha}\xi \Rightarrow ds = x^{\alpha}\, d\xi$$

change of variable

$$y = \int_{C_i} e^{sx} e^{-s^7/7}\, ds = \int_{\tilde{C}_i} x^{\alpha} e^{x^{(\alpha+1)}\xi} e^{-x^{7\alpha}\xi^7}\, d\xi \qquad 7\alpha = \alpha + 1, \quad \alpha = 1/6$$

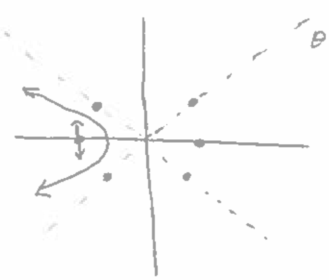
$$= \int_{\tilde{C}_i} e^{x^{7/6}\overbrace{\left(\xi - \frac{\xi^7}{7}\right)}^{h(\xi)}}\, d\xi \sim \underline{\ ?\ } \quad \text{as } |x| \to \infty$$

$$h(\xi) = \xi - \frac{\xi^7}{7}$$

$$h'(\xi) = 1 - \xi^6 = 0 \iff \xi_k = e^{\frac{2\pi i}{6}k} \qquad k = 0, \dots, 5 \qquad \text{Choose } \xi^* = -1$$

$$h''(\xi) = -6\xi^5 \neq 0 \quad \forall \xi_k \quad \Rightarrow \quad n = 2 \ \# \text{ of SD directions}$$

$$h''(-1) = -6(-1)^5 = 6e^{i0} \Rightarrow \alpha = 0$$



Need $\operatorname{Im}(h(\xi)) = 0$ and $\operatorname{Re}(h(\xi)) < 0$

$\Rightarrow \sin(\alpha + 2\theta_m) = 0$ and $\cos(\alpha + 2\theta_m) < 0$

$$\alpha + 2\theta_m = m\pi, \quad \theta_m = \frac{m\pi}{2}$$

$\theta_0: \cos(0) = 1 \not< 0$

$\theta_1: \cos(\pi) = -1 < 0$

$$\theta_{SD} = \frac{\pi}{2}$$

$$\int_{\tilde{C}_4} x^{1/6} e^{x^{7/6}(h(\xi))} = x^{1/6} \int_{\tilde{C}_4} e^{x^{7/6}(h(\xi))}\, d\xi$$

$$\sim x^{1/6} e^{x h(-1)} \sqrt{\frac{2\pi}{x^{7/6}|h''(-1)|}}\, e^{i\pi/2}$$

$$\boxed{y \sim i x^{1/6} e^{-6/7\, x} \sqrt{\frac{\pi}{3x^{7/6}}} \quad \text{as } x \to \infty}$$

[redacted]
Aug 2014

Problem 8

a) Let $\Omega \subset \mathbb{C}$, $x_0 \in \Omega$ and f_n be a sequence of funcs.

Ptws: We say $f_n \to f$ conv ptws if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ st $\forall n > N$ we get $|f_n(x_0) - f(x_0)| < \varepsilon$.

Note: We are only concerned with the fixed pt x_0 and N can vary as ε changes.

Asymp: Fix N. We say $f_N \to f$ conv asymp'ly if $\forall \varepsilon > 0$ $\exists \delta > 0$ st $|x - x_0| < \delta \Rightarrow |f_N(x) - f(x)| < \varepsilon$.

Note We are only concerned w/ the nbhd of x_0, not x_0 itself and N is fixed.

Ex: In each example I'll use Ratio test to show conv.

i) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \left| \frac{x}{n+1} \right| \xrightarrow[n \to \infty]{x \text{ fixed}} 0 \quad \therefore \text{ ptws conv}$$

$$\xrightarrow[n \text{ fixed}]{x \to \infty} \infty \quad \therefore \text{ not asymp conv}$$

$\sum_{n=0}^{\infty} \frac{n!}{x^n}$

$$\left| \frac{(n+1)!}{x^{n+1}} \cdot \frac{x^n}{n!} \right| = \left| \frac{n+1}{x} \right| \xrightarrow[n \to \infty]{x \text{ fixed}} \infty \quad \therefore \text{ not ptw conv}$$

$$\xrightarrow[n \text{ fixed}]{x \to \infty} 0 \quad \therefore \text{ asymp conv}$$

b) Asymptotic expansions are not unique so the mapping is not one to one. However, you can say a function is asymptotic to itself (though that is not necessarilly helpful) so it is an onto mapping. Clearly, this means the mapping is not bijective.

c) Sps $f \sim \tilde{f}$ and $g \sim \tilde{g}$ as $z \to z_0$. We can say $f + g \sim \tilde{f} + \tilde{g}$ as $z \to z_0$ so $\sim$ is distributive. It's important to notice that both asymptotic relations occur as $z \to z_0$, z cannot approach different values for $f \sim \tilde{f}$ and $g \sim \tilde{g}$ or we won't have distributivity.