METHODS OF APPLIED MATHEMATICS COMPREHENSIVE EXAMINATION JANUARY 2018

Work on as many of the following problems as possible. Turn in *all* your work.

(1) Consider the mass-spring system consisting of a mass M_1 attached to a fixed wall by a spring with Hooke's constant k_1, and a second mass M_2 attached to M_1 by a second spring with constant k_2. The masses are constrained to move along the x-axis, with the wall at $x = 0$ and position of M_1 and M_2 at $x_1(t)$ and $x_2(t)$, respectively, with $x_2(0) > x_1(0) > 0$.

(a) Write down the Newtonian dynamics according to Hooke's law (linear elasticity). Sketch the solution of the resulting ODE's for the functions $x_1(t)$ and $x_2(t)$.

(b) Non-dimensionalize with respect to the relevant scales of the setup (masses, spring constants and initial displacements), and identify non-dimensional parameters.

(c) Evaluate a perturbation solution in the small non-dimensional parameter $M_2/M_1 \ll 1$ and sketch the asymptotic solutions in this case as the other parameters vary.

(d) Discuss the differences between this setup and that of a sinusoidally forced harmonic oscillator $M\ddot{y} + ky = A\sin(\omega t)$ as the parameters M, k, ω vary over the positive reals. Can you identify regimes for which this simple system approximates the dynamics of the two-mass set up? What happens to energy conservation in either situation? Discuss.

(2) Consider the eigenvalue problem on the half line $x \geq 0$

$$\epsilon y'' - (U(x) + \lambda)y = 0, \qquad y(0) = 0,\ y(x) \to 0, \text{ as } \quad x \to +\infty.$$

with the potential:

$$U(x) = a\,\delta(x - L)$$

where $\delta(x)$ is the Dirac delta function.

(a) Discuss allowable values of the spectrum, λ, and allowable values of the parameter, a, to have a solution.

(b) Calculate the eigenvalue(s) and eigenfunctions with $L > 0$ fixed as $\epsilon \to 0$. How does this result compare with the free space case? Third,

(c) Discuss the case with $L \sim \epsilon^b$, for $b > 0$. Is there a critical distinguished limit?

(3) Consider the boundary value problem

$$\epsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \epsilon u\,,$$

$$u(0, y) = g(y), \quad u(1, y) = 0, \quad u(x, 0) = f(x), \quad u(x, 1) = 0\,,$$

in $(x, y) \in [0, 1] \times [0, 1]$, for some functions $f(x)$, $g(y)$. Find the first term in an asymptotic expansion of the solution as $\epsilon \to 0$. The functions f and g can be taken to be smooth but are otherwise generic (i.e., do not take special values in $[0, 1]$). If you cannot solve any of the differential equations necessary for your procedure, just set them up properly (i.e., state the proper equation with proper initial and/or boundary conditions etc.).

(4) Consider the functions of a complex variable $z = x + iy$

$$f(z) = \sqrt{\sqrt{z}\sinh\sqrt{z}}, \qquad g(z) = z^z, \qquad h(z) = \sqrt{z^4+1}$$

of the complex variable z.

(a) Choose branch cuts for f to be single valued in the complex z plane.

(b) Show that the level set $\text{Im}(g(z)) = 1$ becomes the real axis $y = 0$ as $x \to \infty$. What does this imply for the graph of imaginary part of the function $g(z)$ along the vertical line $x = x_0$, x_0 large?

(c) Can the function h be defined to be single valued in the domain exterior to the circle $|z| = 2$? Discuss.

(5) Find the first term in the asymptotic expansion of

$$f(x) = \int_C \frac{1}{\zeta^5} \exp\left[ix\left(\zeta^4 - \frac{1}{\zeta^4}\right)\right] d\zeta$$

as $x \to +\infty$, where C is a closed contour around the origin.

(6) Solve the following PDE using the method of characteristics

$$\begin{aligned} \frac{\partial u}{\partial t} + f(t)\frac{\partial u}{\partial x} + g(x,t)\frac{\partial u}{\partial y} &= 0 \\ f(t) &= A\cos\omega t \\ g(x,t) &= B\cos\omega t \sin\alpha x \end{aligned}$$

with the step function initial condition, $u(x, y, 0) = H(y)$. Calculate the support set of the jump discontinuity, $y = y(x, t)$, and sketch its evolution.

(7) (a) Find two term asymptotic expansions as $\epsilon \to 0$ for roots of the equation

$$\epsilon z^8 + (z-5)^2 = \epsilon z$$

(analyze carefully the unperturbed root at $z = 5$, and analyze one root coming in from infinity.)

(b) Find a two term asymptotic expansion for the $x = 0$ unperturbed root of the equation

$$x^4 e^{-x^4} = \epsilon$$

(8) (a) Explain the difference between pointwise convergence and asymptotic convergence. Illustrate with the particular example of power series.

(b) Under which conditions can term-by-term differentiation be applied to an asymptotic series?

Problem 1

a) $$\begin{cases} M_1 \ddot{x}_1 = -K_1(x_1 - d_1) + K_2(x_2 - x_1 - d_2) \\ M_2 \ddot{x}_2 = -K_2(x_2 - x_1 - d_2) \end{cases}$$

where d_1 and d_2 are locations of M_1 and M_2 when the springs are at rest.

For simplicity, let $d_1 = 0$ and $d_2 = d$.

$$\begin{cases} M_1 \ddot{x}_1 = -K_1 x_1 + K_2(x_2 - x_1 - d) & (1) \\ M_2 \ddot{x}_2 = -K_2(x_2 - x_1 - d) & (2) \end{cases}$$

b) Let $L = d$ and $T = \sqrt{\frac{M_2}{K_2}}$ and define

$$x_i = L z_i \quad \text{and} \quad t = T\tau \Rightarrow \frac{dt}{d\tau} = T$$

non-dimensional

$$\frac{dx_i}{dt} = L\frac{dz_i}{dt} = \frac{L}{T}\frac{dz_i}{d\tau} = \frac{L}{T}\dot{z}_i$$

$$\frac{d^2x_i}{dt^2} = \frac{d}{dt}\left(\frac{L}{T}\frac{dz_i}{d\tau}\right) = \frac{L}{T^2}\ddot{z}_i$$

$$\begin{cases} M_1 \frac{\cancel{L}}{T^2}\ddot{z}_1 = -K_1 \cancel{L} z_1 + K_2(\cancel{L} z_2 - \cancel{L} z_1 - \cancel{L}) \\ M_2 \frac{\cancel{L}}{T^2}\ddot{z}_2 = -K_2 \cancel{L}(z_2 - z_1 - 1) \end{cases}$$

$$\begin{cases} \ddot{z}_1 = -\overbrace{\dfrac{T^2 K_1}{M_1}}^{=\alpha} z_1 + \overbrace{\dfrac{T^2 K_2}{M_1}}^{=\beta}(z_2 - z_1 - 1) & (3) \\ \ddot{z}_2 = -(z_2 - z_1 - 1). & (4) \end{cases}$$

Non-Dim Parameters: $\alpha = \dfrac{T^2 K_1}{M_1} = \dfrac{M_2 K_1}{M_1 K_2} = \varepsilon K$; $\beta = \dfrac{T^2 K_2}{M_1} = \dfrac{M_2}{M_1} = \varepsilon$

Both non-dim

c) $M_2 \ll M_1$

$$z_1 = \qquad\qquad = \frac{\ddot{z}_2}{T^2} + z_2 - 1$$

Sub into (3):

$$\frac{\ddddot{z}_2}{T^2} + \ddot{z}_2 = -\varepsilon K(\ddot{z}_2 + z_2 - 1) + \varepsilon\left(\cancel{z_2} - \frac{\ddot{z}_2}{T^2} - \cancel{z_2} + \cancel{1} - \cancel{1}\right)$$

$$\ddddot{z}_2 + \ddot{z}_2\left(1 + \frac{T^2 K_1}{M_1} + \qquad\right) + z_2\left(\frac{T^2 K_1}{M_1}\right) = -\frac{T^2 K_1}{M_1} \qquad (5)$$

Let $k = \dfrac{k_1}{k_2}$, $\varepsilon = \dfrac{M_2}{M_1}$ ∵ $M_2 \ll M_1$. Then (5) becomes

$$\ddddot{z}_2 + \ddot{z}_2(1 + \varepsilon K + \varepsilon) + z_2 \varepsilon K = -\varepsilon K$$

Ansatz: $z_2 = \bar{z} + \varepsilon z_1 + \dots$

$\mathcal{O}(1) = \ddddot{\bar{z}} + \ddot{\bar{z}} = 0$

$r^4 + r^2 = 0$

$r^2(r^2+1) = 0$

$r = 0, 0, \pm i$

$\bar{z} =$

Problem 1

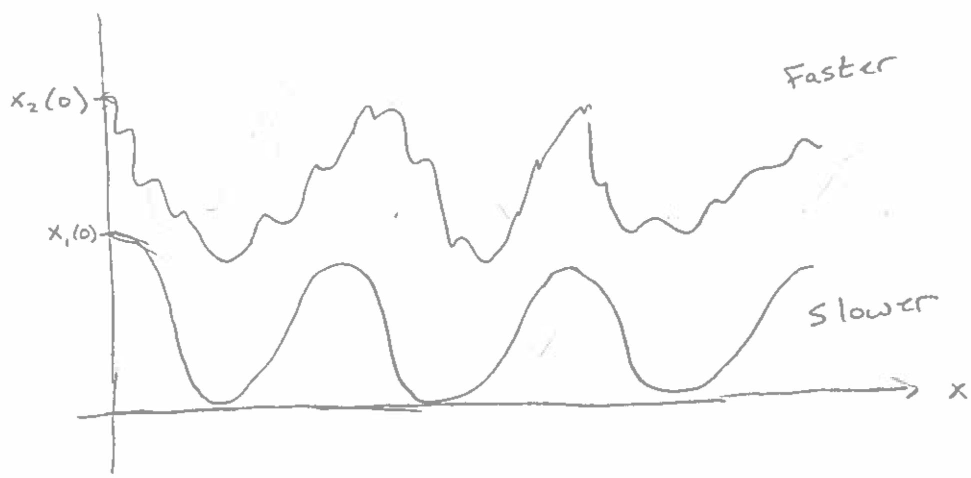
Jan 2018

a)
$$\begin{cases} M_1 \ddot{x}_1 = -k_1(x_1 - d_1) + k_2(x_2 - x_1 - d_2) \\ M_2 \ddot{x}_2 = -k_2(x_2 - x_1 - d_2) \end{cases}$$

where d_1 and d_2 are the locations of M_1 and M_2 when the springs are at rest.

For simplicity, let $d_1 = 0 = d_2$

$$\begin{cases} M_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) & (1) \\ M_2 \ddot{x}_2 = -k_2(x_2 - x_1) & (2) \end{cases}$$



$k_2 > k_1$

b) Let $L = z_1(0)$ and $T = \sqrt{\frac{M_2}{k_2}}$ and define

$x = zL$ and $t = \tau T$ where z and τ are non-dimensional.

$$\dot{x} = \frac{dx}{dt} = \frac{d(Lz)}{dt} = L \frac{dz}{d\tau} \frac{d\tau}{dt} = \frac{L}{T} \dot{z}$$

$$\ddot{x} = \frac{L}{T^2} \ddot{z}$$

$$\Rightarrow \begin{cases} M_1 \frac{\cancel{L}}{T^2} \ddot{z}_1 = -k_1 \cancel{L} z_1 + k_2 \cancel{L} (z_2 - z_1) \\ M_2 \frac{\cancel{L}}{T^2} \ddot{z}_2 = -k_2 \cancel{L} (z_2 - z_1) \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{z}_1 = -\frac{k_1 T^2}{M_1} z_1 + \frac{k_2 T^2}{M_1} (z_2 - z_1) \\ \ddot{z}_2 = -\frac{k_2 T^2}{M_2} (z_2 - z_1) \end{cases}$$

Non-Dim Parameters:

$$\alpha = \frac{k_1 T^2}{M_1} = \frac{M_2 k_1}{M_1 k_2} = \varepsilon K \qquad \beta = \frac{k_2 T^2}{M_1} = \frac{M_2}{M_1} = \varepsilon \qquad \gamma = \frac{k_2 T^2}{M_2} = 1$$

$$\begin{cases} \ddot{z}_1 = -\varepsilon K z_1 + \varepsilon (z_2 - z_1) & (3) \\ \ddot{z}_2 = -(z_2 - z_1) & \end{cases}$$

(4)

c) If $\varepsilon = \frac{M_2}{M_1} \ll 1$, it is our small parameter.

Anzats:

$$z_i = z_i^{(0)} + z_i^{(1)} \varepsilon + \dots$$

Then:

$$\Theta(1): \begin{cases} \ddot{z}_1^{(0)} = 0 \\ \ddot{z}_2^{(0)} = -z_2^{(0)} + z_1^{(0)} \end{cases}$$

$$\dot{z}_1^{(0)}(0) = a = 0$$

$$z_1^{(0)}(0) = b = z_1(0)$$

$$z_1^{(0)} = z_1(0)$$

$$\ddot{z}_2^{(0)} = -z_2^{(0)} + z_1(0)$$

$$z_2^{(0)} = c_1 \cos(\tau) + c_2 \sin(\tau) + z_1(0)$$

Initial vel $= 0$ so $c_2 = 0$

$$\Rightarrow z_2^{(0)} = (z_2(0) - z_1(0)) \cos(\tau) + z_1(0)$$

$$\Theta(\varepsilon): \begin{cases} \ddot{z}_1^{(1)} = -K z_1(0) + (z_2(0) - z_1(0)) \cos(\tau) \\ \ddot{z}_2^{(1)} = -(z_2(0) - z_1(0)) \cos(\tau) \end{cases}$$

$$\begin{cases} \dot{z}_1^{(1)} = -K z_1(0)\tau + (z_2(0) - z_1(0)) \sin(\tau) + \cancel{c_1}^{\,0} \\ \dot{z}_2^{(1)} = -(z_2(0) - z_1(0)) \sin(\tau) + \cancel{c_2}^{\,0} \end{cases}$$

$$\begin{cases} z_1^{(1)} = -\frac{K z_1(0)}{2}\tau^2 - (z_2(0) - z_1(0)) \cos(\tau) + z_2(0) - z_1(0) \\ z_2^{(1)} = (z_2(0) - z_1(0)) \cos(\tau) - (z_2(0) - z_1(0)) \end{cases}$$

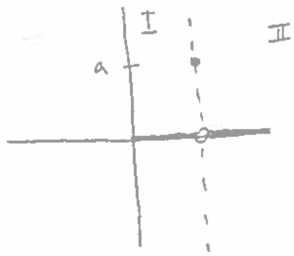
$$\boxed{\begin{aligned} z_1 &\sim z_1 + \varepsilon\left(-\frac{k\, z_1(0)}{2}\tau^2 - (z_2(0) - z_1(0))\cos(\tau) + (z_2(0) - z_1(0))\right) \\ z_2 &\sim (z_2(0) - z_1(0))\cos(\tau) + z_1(0) + \varepsilon\left((z_2(0) - z_1(0))\cos(\tau) - (z_2(0) - z_1(0))\right) \end{aligned}}$$

d)

Problem 2

Jan 2018

$$\begin{cases} \varepsilon y'' - (u(x) + \lambda) y = 0 \\ y(0) = 0 \\ y(x) \to 0 \quad \text{as} \quad x \to \infty \end{cases} \quad \Rightarrow \quad y'' = \frac{\lambda + u(x)}{\varepsilon} y$$



$$u(x) = \alpha \delta(x - L)$$

a) To ensure $y \to 0$ as $x \to \infty$ we need $\frac{\lambda + u(x)}{\varepsilon} > 0$ in region II. So $\frac{\lambda}{\varepsilon} > 0 \Rightarrow \lambda > 0$ given $\varepsilon > 0$. As for the value a, we only need it to be a finite negative value

b) Region I & II: $\quad y'' = \underbrace{\frac{\lambda}{\varepsilon}}_{>0} y$

$$y_i = A_i e^{\sqrt{\lambda/\varepsilon}\, x} + B_i e^{-\sqrt{\lambda/\varepsilon}\, x}$$

Since $y \to 0$ as $x \to \infty$ we know $A_2 = 0$.

$$\Rightarrow \quad y_2 = B e^{-\sqrt{\lambda/\varepsilon}\, x}$$

Since $y(0) = 0$ then

$$y_1(0) = A_1 + B_1 = 0$$
$$B_1 = -A_1 = -A$$

$$\Rightarrow \quad y_1 = A e^{\sqrt{\lambda/\varepsilon}\, x} - A e^{-\sqrt{\lambda/\varepsilon}\, x}$$

Patching:

$$y_1(L) = y_2(L)$$

$$A e^{L\sqrt{\lambda/\varepsilon}} - A e^{-L\sqrt{\lambda/\varepsilon}} = B e^{-L\sqrt{\lambda/\varepsilon}}$$

$$B = A\left(e^{2L\sqrt{\lambda/\varepsilon}} - 1\right)$$

Let $0 < \gamma \ll 1$

$$\int_{L-\gamma}^{L+\gamma} y'' = \int_{L-\gamma}^{L+\gamma} \frac{\lambda + U(x)}{\varepsilon} y$$

$$= \underbrace{\cancel{\int_{L-\gamma}^{L+\gamma} \frac{\lambda}{\varepsilon} y}}_{\text{as } \gamma \to 0} \to 0 + \int_{L-\gamma}^{L+\gamma} \frac{U(x)}{\varepsilon} y$$

$$y'' \Big|_{L-\gamma}^{L+\gamma} = \frac{a}{\varepsilon} y(L)$$

as $\gamma \to 0$: $\quad y_2'(L) - y_1'(L) = \frac{a}{\varepsilon} y(L)$

$$-B\sqrt{\tfrac{\lambda}{\varepsilon}}\, e^{-L\sqrt{\lambda/\varepsilon}} - \left(A\sqrt{\tfrac{\lambda}{\varepsilon}}\, e^{L\sqrt{\lambda/\varepsilon}} + A\sqrt{\tfrac{\lambda}{\varepsilon}}\, e^{-L\sqrt{\lambda/\varepsilon}}\right) = \frac{a}{\varepsilon}\left(A e^{L\sqrt{\lambda/\varepsilon}} - A e^{-L\sqrt{\lambda/\varepsilon}}\right)$$

$$\sqrt{\lambda/\varepsilon}\left(-e^{L\sqrt{\lambda/\varepsilon}} + \cancel{e^{-L\sqrt{\lambda/\varepsilon}}} - e^{L\sqrt{\lambda/\varepsilon}} - \cancel{e^{-L\sqrt{\lambda/\varepsilon}}}\right) = \frac{a}{\varepsilon}\left(e^{L\sqrt{\lambda/\varepsilon}} - e^{-L\sqrt{\lambda/\varepsilon}}\right)$$

$$-2\sqrt{\tfrac{\lambda}{\varepsilon}}\, e^{L\sqrt{\lambda/\varepsilon}} = \frac{a}{\varepsilon}\left(e^{L\sqrt{\lambda/\varepsilon}} - e^{-L\sqrt{\lambda/\varepsilon}}\right)$$

$$\sqrt{\lambda/\varepsilon} = -\frac{a}{2\varepsilon}\left(1 - e^{-2L\sqrt{\lambda/\varepsilon}}\right)$$

Solve for λ

Could maybe do it asymptotically by multiplying by ε.

c) $L \sim \varepsilon^b$ $\quad b > 0$

$$\sqrt{\lambda/\varepsilon} = -\frac{a}{2\varepsilon}\left(1 - e^{-2\sqrt{\lambda}\,\varepsilon^{b-1/2}}\right)$$

$$\sqrt{\lambda\varepsilon} = a\left(e^{-2\sqrt{\lambda}\,\varepsilon^{b-1/2}} - 1\right)$$

If $b > 1/2$ our solution exists.

If $b < 1/2$ you essentially get $0 = -a$ which is not always true.

If $b = 1/2$ then

$$\sqrt{\lambda/\varepsilon} = -\frac{a}{2\varepsilon}\left(1 - e^{0}\right) = 0$$

$\Rightarrow \lambda = 0$

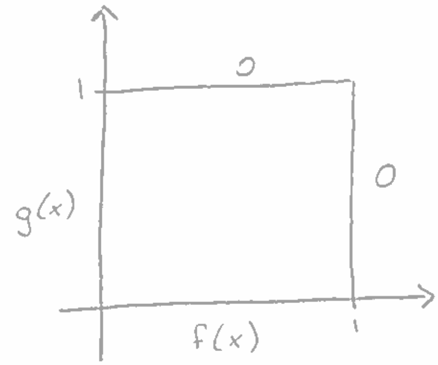
which is an unacceptable value for λ as by part a for non-trivial solns.

Aka: If $b = 1/2$, $y = 0$.

Jan 2018

Problem 3

$$\begin{cases} \varepsilon u_{xx} + u_{yy} = \varepsilon u & (1) \\ u(0,y) = g(y) \quad u(x,0) = f(x) \\ u(1,y) = u(x,1) = 0 \end{cases}$$



Outter

Consider the unpreturbed problem:

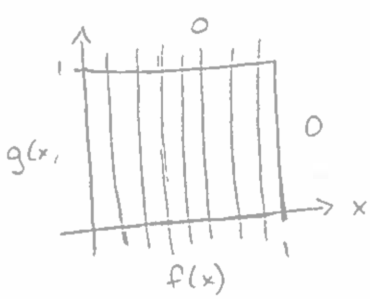
$$u_{yy} = 0$$

$$u = a(x)\,y + b(x)$$

$a(x)$ and $b(x)$ depend on which initial conditions need to be met which depend on the BL location.

Boundary Layer

The characteristic gives us that u_{out} will propogate along the familly $x = c$.



$$\chi = \frac{x - x_0}{\varepsilon^{\alpha}} \Rightarrow \frac{\partial}{\partial x} \mapsto \frac{1}{\varepsilon^{\alpha}}\frac{\partial}{\partial \chi}$$

$$(1) \Rightarrow \frac{\varepsilon}{\varepsilon^{2\alpha}} u_{\chi\chi} + u_{yy} = \varepsilon u$$

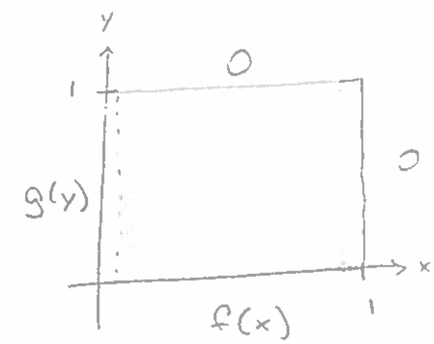
$$1 - 2\alpha = 0$$
$$\alpha = 1/2$$

$$u_{\chi\chi} + u_{yy} = \varepsilon u \qquad ?!$$

Problem 3

Jan 2018

$$\begin{cases} \varepsilon u_{xx} + u_{yy} = \varepsilon u \\ u(0,y) = g(y) \qquad u(x,0) = f(x) \\ u(1,y) = u(x,1) = 0 \end{cases}$$



Outter:

Ansatz: $u = \bar{u} + \varepsilon \bar{u}_1 + \varepsilon^2 \bar{u}_2 + \dots$

$O(1)$: $\bar{u}_{yy} = 0$

$$\bar{u} = a(x)y + b(x)$$

$$\bar{u}(x,1) = a(x) + b(x) = 0 \Rightarrow a(x) = -b(x)$$

$$\bar{u}(x,0) = b(x) = f(x) \Rightarrow \bar{u}(x,y) = -f(x)y + f(x)$$

$$\bar{u}(1,y) = -f(1)y + f(1) = 0$$

$$\bar{u}(0,y) = -f(0)y + f(0) = g(y) \quad \text{Not true } \forall y$$

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Inner:

Two boundary layers $x=0$, $x=1$

$z = \frac{x}{\varepsilon^\alpha}$ near $x=0$

$$\Rightarrow \varepsilon^{1-2\alpha} u_{zz} + u_{yy} = \varepsilon u$$

Balance:

$1-2\alpha = 0 \Rightarrow \alpha = 1/2$

$1-2\alpha = 1 \Rightarrow \alpha = 0$ inconsistent

$0 = 1$ ✗

$$u_{zz} + u_{yy} = \varepsilon u$$

$$u = \tilde{u} + \varepsilon \tilde{u}_1 + \dots$$

$O(1)$: $\tilde{u}_{zz} + \tilde{u}_{yy} = 0$

Matching

$$\lim_{x\to 0} \bar{u} = \lim_{z\to\infty} \tilde{u}$$

$\downarrow$

c

$$u_{uniform} = \bar{u} + \tilde{u} - c$$

Jan 2018

Problem 4

a) $f(z) = \sqrt{\sqrt{z}\sinh\sqrt{z}}$

Branch points:

1) $z = 0$: of $\sqrt{z}$

2) $\sinh(\sqrt{z}) = 0$

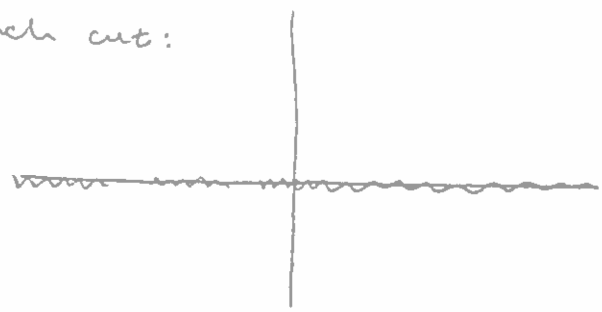
$\sin(i\sqrt{z}) = 0$

$\sin(\sqrt{-z}) = 0$

$\sqrt{-z} = n\pi \qquad n \in \mathbb{Z}$

$z = -(n\pi)^2 = 0, -\pi^2, -4\pi^2, -9\pi^2, \ldots$

Branch cut:



b) $g(z) = z^z \qquad$ Let $z = x + iy = re^{i\theta}$

$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}(\frac{y}{x})$

$z^z = (re^{i\theta})^{x+iy} = r^x e^{-y\theta} r^{iy} e^{i\theta x}$

$= r^x e^{-y\theta}(e^{i(y\ln r + \theta x)})$

$\mathrm{Im}(z^z) = r^x e^{-y\theta}\sin(y\ln r + \theta x) = 1$

$e^{-y\tan^{-1}(\frac{y}{x})}\sin\left(\frac{y}{2}\ln(x^2+y^2) + x\tan^{-1}(\frac{y}{x})\right) = \frac{1}{(x^2+y^2)^{x/2}}$

as $x \to \infty$

$e^{-y\tan(\frac{y}{x})} \to 1 \qquad x\tan^{-1}(\frac{y}{x}) \to 0 \qquad \frac{1}{(x^2+y^2)^{x/2}} \to 0$

So we are essentially left with:

$$\lim_{x \to \infty} \sin\left(\frac{y}{2} \ln(x^2+y^2)\right) = 0$$

$$\Rightarrow \lim_{x \to \infty} \left(\frac{y}{2} \ln(x^2+y^2)\right) = 0$$

$\Rightarrow$ $y \to 0$ to make the above line true.

c) $h(z) = \sqrt{z^4+1}$

BP: $z^4 + 1 = 0$

$$z = e^{i\pi/4},\ e^{i3\pi/4},\ e^{i5\pi/4},\ e^{i7\pi/4}$$

All branch points are inside the region $|z| = 2$ and all branch cuts can be made inside that region as well. Thus, removing $|z| < 2$ from the domain removes <u>all</u> branch points and cuts b/c ∞ is not a bp.

$$z = \frac{1}{s} \Rightarrow \sqrt{\frac{1}{s^4}+1} = \frac{\sqrt{1+s^4}}{\sqrt{s^4}} \qquad e^{4i\theta} \Rightarrow e^{2i\theta}$$

$$e^{4i\theta+8\pi} \Rightarrow e^{2i\theta+4\pi} = e^{2i\theta} \Rightarrow \infty \text{ not b}$$

For the domain outside $|z| = 2$, to wind around 1 bp requires winding around all 4 which will make $h(z)$ single valued.

Problem 5

Jan 2018

$$I(x) = \int_C \frac{1}{s^5} e^{ix\left(s^4 - \frac{1}{s^4}\right)} ds$$

No branch points, just a pole of order 5 at $s=0$.

Let $s = e^{i\theta} \Rightarrow ds = ie^{i\theta} d\theta$ where $\theta \in [0, 2\pi)$

$$I(x) = i\int_0^{2\pi} \frac{1}{e^{5i\theta}} e^{ix\left(e^{i4\theta} - e^{-i4\theta}\right)} \cdot e^{i\theta} d\theta$$

$$= i\int_0^{2\pi} e^{-4\theta i} e^{ix\left(e^{4\theta i} - e^{-4\theta i}\right)} d\theta$$

$$\sin\theta = \frac{e^{\theta i} - e^{-\theta i}}{2i}$$

$$= i\int_0^{2\pi} \cos(4\theta) e^{-2x\sin 4\theta} d\theta + \int_0^{2\pi} \sin(4\theta) e^{-2x\sin(4\theta)} d\theta$$

Now that we are defined in and integrating over the reals, we can use Laplace's Method.

$g_1(\theta) = \cos(4\theta)$ $\qquad g_2(\theta) = \sin(4\theta)$

$h(\theta) = -2\sin(4\theta)$

$h'(\theta) = -8\cos(4\theta) = 0$ $\qquad \theta^* = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \ldots, \frac{15\pi}{8}$

same for $\frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

$h\left(\frac{\pi}{8}\right) = -2\sin\left(\frac{\pi}{2}\right) = -2$

$h\left(\frac{3\pi}{8}\right) = -2\sin\left(\frac{3\pi}{2}\right) = 2$

θ_1

same for

$\frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

$\theta_2 \quad \theta_3 \quad \theta_4$

$\theta_1, \theta_2, \theta_3, \theta_4$ all have equal weight.

$$h''(\theta) = 32 \sin(4\theta)$$

$$h''(\theta_i) = -32$$

$$f(x) \sim \left(i g_1\left(\tfrac{3\pi}{2}\right) + g_2\left(\tfrac{3\pi}{2}\right)\right) e^{x h\left(\frac{3\pi}{2}\right)} \sqrt{\frac{2\pi}{x \left|h''\left(\frac{3\pi}{2}\right)\right|}} + \text{3 terms of equal weight}$$

as $x \to \infty$

$g_1(\theta_1) = \cos\left(\frac{3\pi}{2}\right) = 0$ $\quad$ $g_2(\theta_1) = \sin\left(\frac{3\pi}{2}\right) = -1$

$g_1(\theta_2) = \cos\left(\frac{7\pi}{2}\right) = 0$ $\quad$ $g_2(\theta_2) = -1$

$g_1(\theta_3) = \cos\left(\frac{11\pi}{2}\right) = 0$ $\quad$ $g_2(\theta_3) = -1$

$g_1(\theta_4) = 0$ $\quad$ $g_4(\theta_4) = -1$

$$f(x) \sim \left(\sum_{i=1}^{4} g_2(\theta_i) \right) e^{2x} \sqrt{\frac{2\pi}{32x}} \quad \text{as } x \to \infty$$

$$f(x) \sim -4 e^{2x} \sqrt{\frac{\pi}{16x}} \quad \text{as } x \to \infty$$

$$\boxed{f(x) \sim -e^{2x} \sqrt{\frac{\pi}{x}} \quad \text{as } x \to \infty}$$

Problem 6

Jan 2018

$$\begin{cases} u_t + f(t)\,u_x + g(x,t)\,u_y = 0 \\ f(t) = A\cos(\omega t) \\ g(x,t) = B\cos(\omega t)\sin(\alpha x) \\ u(x,y,0) = u_0(x,y) = H(y) \end{cases} \tag{1}$$

Define the following:

$$\begin{cases} z(s) = u\big(X(s), Y(s), T(s)\big) \Rightarrow z_s = u_x \frac{dX}{ds} + u_y \frac{dY}{ds} + u_T \frac{dT}{ds} & (2) \\ X(0) = x \\ Y(0) = y \\ T(0) = t \end{cases}$$

$z_s = 0$ by (1). Also by (1) and (2) we get

$$\begin{cases} \frac{dX}{ds} = f(T(s)) \\ \frac{dY}{ds} = g(X(s), T(s)) \\ \frac{dT}{ds} = 1 \quad \Rightarrow \quad T(s) = s + t \end{cases}$$

$$\frac{dX}{ds} = A\cos(\omega(s+t)) \quad \Rightarrow \quad X(s) = \frac{A}{\omega}\sin(\omega(s+t))$$

$$\frac{dY}{ds} = B\cos(\omega s + \omega t)\sin\left(\frac{A\alpha}{\omega}\sin(\omega s + \omega t)\right) \tag{3}$$

Solve for $Y(s)$.

$\Rightarrow z(s) = z(0) \qquad \because z_s = 0$

Note: $z(0) = u(X(0), Y(0), T(0)) = u(x, y, t)$

Let $s = -t$ then $z(-t) = u_0(X(-t), Y(-t)) = H(Y(-t))$

$\Rightarrow z(0) = Z(-t)$

$u(x,y,t) = H(Y(-t))$

↳ where you get Y by solving (3)

Problem 7

$$\varepsilon z^8 + (z-5)^2 = \varepsilon z \qquad (1)$$

a) Consider the unpreturbed problem:

$$(z-5)^2 = 0$$

$$z = 5$$

$$\Rightarrow \quad z_0 = 5 + a\varepsilon^{1/2}$$

Plug z_0 into (1):

$$\varepsilon(5 + a\varepsilon^{1/2})^8 + (a\varepsilon^{1/2})^2 = \varepsilon(5 + a\varepsilon^{1/2})$$

$$O(\varepsilon): \quad 5^8 + a^2 = 5$$

$$a = \pm\sqrt{5 - 5^8}$$

$$\boxed{\begin{aligned} z_1 &\sim 5 + \varepsilon i\sqrt{5^8 - 5} \\ z_2 &\sim 5 - \varepsilon i\sqrt{5^8 - 5} \end{aligned}}$$

Let $x = \varepsilon^{\alpha} z \Rightarrow z = \varepsilon^{-\alpha} x$. Sub this into (1):

$$\varepsilon^{1-8\alpha} x^8 + \varepsilon^{-2\alpha} x^2 + 2\varepsilon^{-\alpha} x + 25 = \varepsilon^{1-\alpha} x$$

$$1 - 8\alpha = -2\alpha$$

$$\alpha = 1/6$$

$$\Rightarrow \quad \varepsilon^{-1/3} x^8 + \varepsilon^{-1/3} x^2 + 2\varepsilon^{-1/6} x + 25 = \varepsilon^{5/6} x$$

$$x^8 + x^2 + 2\varepsilon^{1/6} x + 25\varepsilon^{1/6} = \varepsilon x \qquad (2)$$

Again we consider the unpreturbed problem:

$$x^8 + x^2 = 0$$

$$x = 0, 0, \omega, \ldots, \omega^6 \qquad \omega^6 = -1$$

correspond to z_1 and z_2

$x_i = \omega^i + b_1^i \varepsilon^{1/6} + b_2^i \varepsilon^{2/6} + \dots$ $\quad i = 1, \dots, 6$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21
1 8

Sub into (2)

$O(1)$: $\omega^{8i} + \omega^{2i} = 0$ ✓

$O(\varepsilon^{1/6})$: $8\omega^{7i} b_1^i + 2\omega^i b_1^i + 2\omega^i + 25$

$$b_1^i = \frac{-2\omega^i + 25}{8\omega^{7i} + 2\omega^i} = \frac{25 - 2\omega^i}{2\omega^i - 8\omega^i}$$

$\Rightarrow x_i \sim \omega^i + b_1^i \varepsilon^{1/6}$ where b_1^i comes from

$z_{i+2} = \varepsilon^{-1/6} x_i$ from earlier assumption

$$\boxed{z_{i+2} \sim \frac{\omega^i}{\varepsilon^{1/6}} + b_1^i}$$ $\quad i = 1, \dots, 6$

b) $x^4 e^{-x^4} = \varepsilon$ (3)

$$x^4 = \varepsilon e^{x^4} \sim \varepsilon(1 + x^4)$$

Unpreturbed roots are: $x = 0$

$$x_0 = \sum_{n=1}^{\infty} a_n \varepsilon^{n/4}$$

Plug x_0 into (3):

$$\left(a_1 \varepsilon^{1/4} + \cancel{a_2 \varepsilon^{1/2}} + \dots\right)^4 = \varepsilon + \varepsilon\left(a_1 \varepsilon^{1/4} + \cancel{a_2 \varepsilon^{1/2}} + \dots\right)^4$$

$O(\varepsilon)$: $a_1^4 = 1$

$$a_1 = \pm i, \pm 1$$

$O(\varepsilon^{5/4})$: $4 a_1^3 a_2 = 0$

$$a_2 = 0$$

$O(\varepsilon^{6/4})$: $4 a_1^3 a_3 = 0$

$$a_3 = 0$$

$O(\varepsilon^{7/4})$: $4 a_1^3 a_4 = 0$

$$a_4 = 0$$

$O(\varepsilon^2)$: $4 a_1^3 a_5 = a_1^4$

$$a_5 = \frac{a_1}{4}$$

$$\boxed{\begin{aligned} x_1 &\sim \varepsilon^{1/4} + \tfrac{1}{4}\varepsilon^{5/4} \\ x_2 &\sim -\varepsilon^{1/4} - \tfrac{1}{4}\varepsilon^{5/4} \end{aligned}} \quad \text{as } \varepsilon \to 0$$

$$\left.\begin{aligned} x_3 &\sim -i\varepsilon^{1/4} - \tfrac{i}{4}\varepsilon^{5/4} \\ x_4 &\sim i\varepsilon^{1/4} + \tfrac{i}{4}\varepsilon^{5/4} \end{aligned}\right\} \text{ no imaginary roots}$$

Problem 8

a) Let $\Omega \subset \mathbb{C}$, $x_0 \in \Omega$ and f_n be a seq of funcs.

Ptws: We say $f_n \longrightarrow f$ conv ptws if $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ st $\forall n > N \Rightarrow |f_n(x_0) - f(x_0)| < \varepsilon$.

Note: We are only concerned with fixed x_0 and N can change.

Asymp: For fixed N, we say $f_N \rightarrow f$ conv asymp'ly if $\forall \varepsilon > 0 \; \exists \delta > 0$ st $|x - x_0| < \delta \Rightarrow |f_N(x) - f(x)| < \varepsilon$.

Note: This time N is fixed and x can change.

Ex: In each example I'll use ratio test to show convergence.

i) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \left| \frac{x^{n+1}}{n+1} \right| \xrightarrow[x \to \infty]{n \text{ fixed}} \infty \quad \therefore \text{ not asymp conv}$$

$$\xrightarrow[x \text{ fixed}]{n \to \infty} 0 \quad \therefore \text{ ptws conv}$$

ii) $\sum_{n=0}^{\infty} \frac{n!}{x^n}$

$$\left| \frac{(n+1)!}{x^{n+1}} \cdot \frac{x^n}{n!} \right| = \left| \frac{n+1}{x} \right| \xrightarrow[x \to \infty]{n \text{ fixed}} 0. \quad \therefore \text{ asymp conv}$$

$$\xrightarrow[x \text{ fixed}]{n \to \infty} \infty \quad \therefore \text{ not ptws conv}$$

b) When the function is uniformly continuous