METHODS OF APPLIED MATHEMATICS COMPREHENSIVE EXAMINATION AUGUST 2013

Work on as many of the following problems as possible. Turn in *all* your work.

(1) Consider a projectile of mass m fired vertically from the surface of the Earth at one location on the equator. Assume constant gravity g along the radial direction and neglect any air drag, but do account for the Earth rotation.

(a) Write down the equation of motion of the projectile in the equatorial plane with initial velocity, with respect to the rotating Earth system, $\mathbf{v} = v\mathbf{r}$, where $\mathbf{r}$ is the unit vector in the radial direction from the Earth's center to the firing location.

(b) Under the problem's assumptions, does the projectile always return to Earth?

(c) If the projectile returns to Earth, compute the return location, based on the first term in an asymptotic expansion for $v \ll \omega R$, where ω and R are the Earth's angular velocity and radius, respectively.

(2) Find a Laplace transform solution of the following boundary value problem

$$u_{tt} - u_{xx} + 2u_x = 0\,, \qquad u(x,0) = u_t(x,0) = 0\,, \qquad u(0,t) = t^2$$

for the unknown function $u(x,t)$ defined on the half-line $x \geq 0$, with $u(x,\cdot) \to 0$ for physical relevance of the solution. Discuss the asymptotic values of the solution for large values of x and t, for both cases $x > t$ and $x < t$.

(3) For small values of the parameter $\epsilon > 0$, find the leading order term in the asymptotic expansion valid for long times of the solution to the following ODE

$$\frac{d^2x}{dt^2} + \left(4 + \sin^2(\epsilon t) - 4\sin(\epsilon t)\right)x = 0\,, \qquad x(0) = 1\,, \quad \frac{dx}{dt}(0) = 0.$$

(4) Consider the following "quarter-plane" boundary value problem for the diffusion equation with constant diffusivity κ,

$$c_t = \kappa\, c_{xx}\,, \qquad c(x,0) = 0\,, \qquad c(0,t) = t^2$$

(a) Use the Laplace transform *in space* (!),

$$C(s,t) \equiv \int_0^\infty e^{-sx}\, c(x,t)\, dx\,, \qquad \mathrm{Re}\, s > 0\,,$$

to find a solution for all times $t > 0$ in the half line $x \geq 0$.

(b) Discuss the limit $x \to 0^+$ and how to satisfy the boundary condition.

(5) Consider the function of the complex variable z

$$w = \log \frac{z+i}{z-i}$$

(a) Compute the real and imaginary part of w.
(b) Discuss the singularities and propose a domain where w can be a single valued analytic function of z.
(c) Under which branch choices is $\operatorname{Im} w = 0$ and $\operatorname{Re} w > 0$ for $\operatorname{Im} z = 0$ and $\operatorname{Re} z > 0$? Discuss.

(6) The method of images can be used to enforce boundary conditions for Laplace's equation in regions of the plane $\mathbb{R}^2$. Consider the function

$$w = z + 1/z$$

(a) Sketch the level sets $\operatorname{Im} w =$ const., in the plane with Cartesian coordinates (x, y), with $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. What happens when const.$= 0$?
(b) Can you propose a choice for the location of a set of simple poles (images)

$$\frac{c_k}{z + d_k}, \qquad c_k, d_k \text{ complex}, \quad k \in \mathbb{N}$$

to be added to the definition of w so that two level sets $\operatorname{Im} w =$ const. corresponds to the horizontal lines $y = \pm 2$?

(7) Consider the function of t defined through the contour integral

$$\frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s^3} e^{t(s+m(1-\sqrt{1+s^2}))}\, ds, \qquad 0 < m < 1, \quad c > 0,$$

(a) State clearly the assumptions on the choice of branch cuts for the contour integral to be well defined, and reduce the contour integral to a real one evaluated along branch cuts.
(b) Evaluate the integral using Laurent expansions (residues).
(c) Find the leading order term in the asymptotic expansion of the integral as $t \to \infty$, m fixed. What happens if $m > 1$? Discuss. How do these results compare with the evaluation by residues? Discuss.

(8) (a) Explain the difference between pointwise convergence and asymptotic convergence. Illustrate with the particular examples of (i) of power series, and (ii) a Laurent series.
(b) Are asymptotic series unique? Explain.
(c) How does an asympotic relation carry over to functional compositions? Discuss in the context of the following example: does

$$f(x) \sim g(x) \qquad \text{as} \quad x \to \infty$$

imply, for a given function F,

$$F[f(x)] \sim F[g(x)] \qquad \text{as} \quad x \to \infty?$$

Make sure to clarify your assumptions on f, g and F for a definitive, but non trivial, statement.

$$\begin{cases} u_{tt} - u_{xx} + 2u_x = 0 & (1) \\ u(x,0) = u_t(x,0) = 0 \\ u(0,t) = t^2 \end{cases}$$

$\mathcal{L}(u) = \hat{u}$

$$\mathcal{L}(u_{tt}) = \int_0^\infty e^{-st} \frac{d^2}{dt^2} u(x,t)\,dt = \cancel{e^{-st} u_t\Big|_0^\infty} + s\int_0^\infty e^{-st}\frac{d}{dt}u(x,t)\,dt$$

$$= \cancel{e^{-st} u(x,t)\Big|_0^\infty} + s^2\int_0^\infty e^{-st} u(x,t)\,dt = s^2\hat{u}$$

$\mathcal{L}(u_{xx}) = \hat{u}_{xx}$

Sub into (1):

$$s^2\hat{u} - \hat{u}_{xx} + 2\hat{u}_x = 0$$

$$r^2 - 2r - s^2 = 0$$

$$r = \frac{2 \pm \sqrt{4+4s^2}}{2} = 1 \pm \sqrt{s^2+1}$$

$$\hat{u} = a(s)e^{x(1-\sqrt{s^2+1})} + b(s)e^{x(1+\sqrt{s^2+1})}$$

For physical relevance as $x \to \infty$ we must set $b(s) = 0$.

$$\hat{u} = a(s)e^{x(1-\sqrt{s^2+1})}$$

$$a(s) = \hat{u}(0,s) = \int_0^\infty e^{-st}t^2\,dt = \cancel{\frac{t^2 e^{-st}}{-s}}\Big|_0^\infty + \frac{2}{s}\int_0^\infty t e^{-st}\,dt$$

$$= \frac{2}{s}\left(\cancel{t\left(-\tfrac{1}{s}\right)e^{-st}}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt\right) = \frac{2}{s^3}$$

$$\hat{u} = \frac{2}{s^3}e^{x(1-\sqrt{s^2+1})}$$

$$u(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\frac{2e^{x(1-\sqrt{s^2+1})}}{s^3}\,ds = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{s^3}e^{st-x(1-\sqrt{s^2+1})}\,ds$$

Wrong

Consider $t = mx \Rightarrow m = \frac{t}{x}$

$$\Rightarrow u(x,t) = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s^3} e^{\underbrace{x(s - m(1-\sqrt{s^2+1}))}_{h(s)}} ds$$

$0 < m < 1 \Rightarrow x > t$
$1 < m \Rightarrow t > x$

$$h'(s) = 1 - \frac{ms}{\sqrt{s^2+1}} = 0$$

$$\sqrt{s^2+1} = ms$$

$$s^2 + 1 = m^2 s^2$$

$$s^2 = \frac{1}{m^2-1}$$

If $0 < m < 1 \Rightarrow$ s.p.'s are imaginary (see prob 7)

If $m > 1 \Rightarrow$ s.p.'s are real

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$$\begin{cases} u_{tt} - u_{xx} + 2u_x = 0 \\ u(x,0) = u_t(x,0) = 0 \\ u(0,t) = t^2 \end{cases} \qquad (1)$$

$\mathcal{L}(u) = \hat{u}$

$$\mathcal{L}(u_{tt}) = \int_0^\infty e^{-st} \frac{d^2}{dt^2} u \, dt = \cancel{e^{-st} u_t \Big|_0^\infty}^{\to 0} + s \int_0^\infty e^{-st} \frac{d}{dt} u \, dt$$

$$= e^{-st} u \Big|_0^\infty + s^2 \int_0^\infty e^{-st} u \, dt = s^2 \hat{u}$$

$\mathcal{L}(u_{xx}) = \hat{u}_{xx}$

Then $\mathcal{L}((1))$ is:

$$s^2 \hat{u} - \hat{u}_{xx} + 2u_x = 0$$

$$\lambda^2 - 2\lambda - s^2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4 + 4s^2}}{2} = 1 \pm \sqrt{1+s^2}$$

$$\hat{u} = a(s) e^{x(1 - \sqrt{1+s^2})} + b(s) e^{x(1+\sqrt{1+s^2})}$$

For the soln to be physically relevant as $x \to \infty$,

we need $b(s) = 0$.

$$\hat{u} = a(s) e^{x(1 - \sqrt{1+s^2})}$$

$$a(s) = \hat{u}(0,s) = \int_0^\infty e^{-st} u(0,t) \, dt = \int_0^\infty t^2 e^{-st} ds$$

$$= \cancel{-\frac{t^2 e^{-st}}{s}}^{\to 0} \Big|_0^\infty + \frac{2}{s} \int_0^\infty t e^{-st} dt$$

$$= \cancel{-\frac{2te^{-st}}{s^2} \Big|_0^\infty}^{\to 0} + \frac{2}{s^2} \int_0^\infty e^{-st} dt = -\frac{2}{s^3} e^{-st} \Big|_0^\infty = \frac{2}{s^3}$$

$$\hat{u} = \frac{2}{s^3} e^{x(1 - \sqrt{1+s^2})}$$

$$u(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\,\frac{2e^{x(1-\sqrt{1+s^2})}}{s^3}\,ds = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st+x(1-\sqrt{1+s^2})}}{s^3}\,ds$$

Consider $x = mt \Rightarrow m = \frac{x}{t}$

$$u(x,t) = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s^3}\, e^{\underbrace{t(s+m(1-\sqrt{1+s^2}))}_{h(s)}}\,ds$$

$$h'(s) = 1 - \frac{ms}{\sqrt{1+s^2}} = 0$$

$$ms = \sqrt{1+s^2}$$

$$m^2 s^2 = 1+s^2$$

$$s^2 = \frac{1}{m^2-1}$$

$$s = \frac{1}{\sqrt{m^2-1}}$$

If $0 \le m < 1$ then s.p. = imaginary (see prob 7)

If $1 \le m$ then s.p. = real

Problem 7

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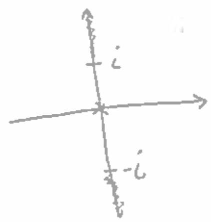
$$I(m,t) = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s^3}\, e^{t(s+m(1-\sqrt{1+s^2}))}\, ds \qquad 0<m<1 \qquad c>0$$

a) $s=0$ triple pole

$s=\pm i$ branch pt

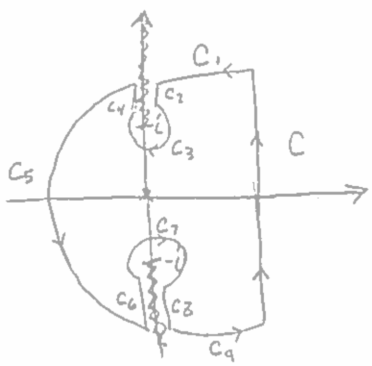
Let $F(s,m) = \dfrac{e^{tm(1-\sqrt{1+s^2})}}{s^3}$

Branch cut:



← domain on which $F(s,m)$ is single-valued.

Pick + branch of $\sqrt{\ }$.



$$\tilde{C} = C + \sum_{i=1}^{9} C_i$$

$$\int_{\tilde{C}} e^{st} F(s,m)\, ds = 2\pi i \operatorname{Res}\left(e^{st}F(s,m),\ s=0\right)$$