METHODS OF APPLIED MATHEMATICS COMPREHENSIVE EXAMINATION JANUARY 2017

Work on as many of the following problems as possible. Turn in *all* your work.

(1) Consider the projectile problem of a body (mass m), launched vertically from the surface of the Earth (assume an inverse square attractive gravitational field and straight line motion). Assuming that the object always experiences a cubic drag force in its velocity:

(a) Write down the Newtonian dynamics.

(b) Non-dimensionalize with respect to kinematic scales (constant gravity, zero drag), identify non-dimensional parameters.

(c) Compute the apex height assuming small initial velocities.

(d) Sketch solution. Is it symmetric about its apex (if it exists)? Explain.

(e) (bonus) Does an apex always exist for all initial velocities for this model? Can you prove it?

(2) Consider the quarter-plane problem $x \geq 0$, $t \geq 0$ for the following PDE:

$$u_t = u_{xx} \qquad u(x,0) = 0 \qquad u(x,L) = 0 \qquad u(0,t) = \sin \omega t\,.$$

(a) Write the solution using the Laplace Transform pair

$$\mathcal{L}(u) = U(x,s) = \int_0^\infty e^{-st}\, u(x,t)\, dt\,, \qquad u(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} U(x,s) ds$$

where the contour for the inverse transform is the usual straight line parallel to the imaginary axis of the s-plane, crossing the real axis at the constant $c > 0$.

(b) Discuss the analytic properties of $U(x,s)$ in the complex s-plane, and classify its singularities. How do things change in the limit as $L \to \infty$

(c) Discuss the long time behavior of the solution $u(x,t)$. How is the behavior different for finite L versus $L \to \infty$

(3) Consider the integral

$$I(a,b) = \int_0^b \frac{dx}{a + i\cos x}$$

(a) Prove that $I(a, 2\pi)$ is real if $a, b \in \mathbb{R}$.

? Typo? (b) Let $x = \exp(iz)$ and evaluate $I(a,b)$, defining all necessary branch cuts to evaluate. Discuss its analyticity if b is complex (hint: partial fractions, and pay attention to using anti-derivatives only where appropriate.)

(c) For what values of b is $I(a,b)$ real?

(4) Consider the rapidly varying diffusivity

$$K(x,\epsilon) = A + G(x,\epsilon) \quad \frac{x}{\epsilon}$$

where A is chosen to guarantee K is positive, and ϵ is a small constant. Also consider the function, $v(x)$. By applying homogenization, average the following advection-diffusion equation

$$\frac{\partial u}{\partial t} + \frac{1}{\epsilon} v(x/\epsilon)\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(K(x,\epsilon)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K(x,\epsilon)\frac{\partial u}{\partial y}\right)$$

$$u(x,y,0) = u_0(x,y)\,,$$

and calculate a leading order effective equation governing the evolution as $\epsilon \to 0$ over the (x, y)-plane, assuming the functions $G(x, \epsilon)$, $v(x)$ are mean zero, periodic, and share the same period. Solve the averaged equation in free space.

(5) Consider the following contour integral

$$I(t; m) = \frac{\omega}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{t(s-m\sqrt{s})}}{\omega^2 + s^2} ds$$

with respect to the real parameters $m > 0, \omega > 0$.

(a) Compute the first term in the asymptotic expansion of I as $t \to \infty$, for fixed parameters. Does this depend on the relative magnitude of m and ω? Discuss.

(b) If $m \equiv x/t$, how does this asymptotic expansion compare with that of problem (2)? Discuss.

(6) (a) Find two term asymptotic expansions as $\epsilon \to 0$ for roots of the equation

$$\epsilon z^5 + (z + 1)^2 = \epsilon$$

(analyze carefully the unperturbed root at $z = -1$, and analyze one root coming in from infinity.)

(b) Find a two term asymptotic expansion for the $x = 0$ unperturbed root of the equation

$$x^2 e^{-x^2} = \epsilon$$

(Bonus): Discuss a strategy for computing the root at infinity.

(7) Consider the following initial value problem for the time-t evolution equation in one spatial dimension $x \in \mathbb{R}$

$$T_t + \gamma x T_y = \kappa(T_{xx} + T_{yy}), \qquad T(x, 0) = T_0(\alpha x, \alpha y).$$

(a) What are the units of the parameters γ, κ and α?

(b) Use the Fourier-transform method with definition

$$\hat{T}(k, \eta) \equiv \int_{R^2} dx dy T(x, t) e^{-i(kx+\eta y)}$$

to solve the resulting equation for $\hat{T}$ by characteristics.

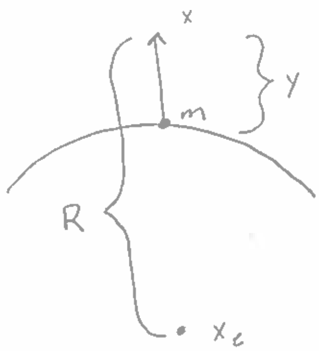
(c) Compute the long-time, pointwise (at fixed (x,y) position) asymptotics of T assuming that $\hat{T}_0(0, 0) \neq 0$.

(8) (a) Explain the difference between pointwise convergence and asymptotic convergence. Illustrate with the particular example of power series.

(b) Given a real integral, $\int_a^b f(x, y) dy$, with $f(x, y) \sim f_0(y)$ as $x \to x_0$, under what conditions is termwise integration guaranteed to yield valid asymptotics?

Problem 1

a)



$$m\ddot{x} = -\frac{GmM}{|x-x_e|^2} + \beta\dot{x}^3$$

$$M\ddot{x}_e = \frac{GmM}{|x-x_e|^2}$$

The velocity of an object is inversely proportional to its mass. Since the mass of the Earth is substantially larger than the projectile's, its velocity will be substantially smaller thus $\dot{x}_e \approx$ const so $\ddot{x}_e \approx 0$. Thus

$$\begin{cases} \ddot{x} = -\dfrac{GM}{|x-x_e|^2} + \dfrac{\beta}{m}\dot{x}^3 \\ x(0) = R_e = \text{radius of Earth} \\ \dot{x}(0) = v_0 > 0 \end{cases}$$

Change of variables

$$\begin{cases} y = x - R_e \quad \Rightarrow \quad x = y + R_e \\ x_e = 0 \end{cases}$$

$$\Rightarrow \quad \ddot{y} = -\frac{GM}{|y+R_e|^2} + \frac{\beta}{m}y^3$$

$$\boxed{\ddot{y} = -\underbrace{\frac{GM}{R_e^2}}_{g}\frac{1}{|1+\frac{R_e}{y}|^2} + \frac{\beta}{m}\dot{y}^3}$$

where $y(0) = x(0) - R_e = 0$

and $\dot{y}(0) = \dot{x}(0) = v_0 > 0$

b) We know $y \ll R_e$ so consider $\frac{y}{R_e} \approx 0$.

$$\Rightarrow \quad \ddot{y} = -g$$

$$\dot{y} = -gt + c \qquad \dot{y}(0) = c \qquad \Rightarrow \quad \dot{y} = -gt + v_0$$

$$y = -\tfrac{1}{2} g t^2 + v_0 t + c \qquad y(0) = 0 \qquad \Rightarrow \quad y = -\tfrac{1}{2} g t^2 + v_0 t$$

At apex: $t = t_a$

$$\dot{y}(t_a) = 0 \quad \Rightarrow \quad t_a = \frac{v_0}{g}$$

$$\ddot{y}(t_a) = -\frac{1}{2} g \left(\frac{v_0}{g}\right)^2 + \frac{v_0^2}{g} = \frac{v_0^2}{2g} \left[\frac{(m/s)^2}{m/s^2}\right]$$

Choose $L = \frac{v_0^2}{g}$ and $T = \frac{v_0}{g}$ and let

$$y = zL \quad \text{and} \quad t = \tau T \quad \Rightarrow \quad \frac{dt}{d\tau} = T$$

non-dimensional

$$\frac{dy}{dt} = \frac{d(zL)}{dt} = L \frac{dz}{dt} = L \frac{dz}{d\tau}\frac{d\tau}{dt} = \frac{L}{T}\frac{dz}{d\tau} = \frac{L}{T} z_\tau = v_0 \dot{z}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{L}{T}\frac{dz}{d\tau}\right) = \frac{L}{T}\frac{d^2z}{d\tau^2}\frac{d\tau}{dt} = \frac{L}{T^2}\frac{d^2z}{d\tau^2} = g\ddot{z}$$

$$\Rightarrow \quad g\ddot{z} = -g \frac{1}{\left|1 + \frac{zL}{R_e}\right|^2} - \frac{\beta}{m}(v_0 \dot{z})^3$$

Let $\varepsilon = \frac{L}{R_e}$ and $\alpha = \frac{\beta v_0 R_e}{m}$

$$\Rightarrow \quad \boxed{\ddot{z} = -\frac{1}{|1 + \varepsilon z|^2} - \varepsilon \alpha \dot{z}^3} \qquad (1)$$

$\because t = 0 \Rightarrow \tau \Rightarrow 0$

$$v_0 = \dot{y}(0) = v_0 \dot{z}(0)$$

$$\Rightarrow \quad \dot{z}(0) = 1$$

$$0 = y(0) = L z(0)$$

$$\Rightarrow \quad z(0) = 0$$

c) For sufficiently small v, we get $\varepsilon \ll 1$. so consider the following:

$$z \sim z_0 + \varepsilon z_1 + \varepsilon^2 z_2 + \dots$$

and plug into (1) to get:

$$\ddot{z}_0 + \varepsilon \ddot{z}_1 + \dots = -\frac{1}{(1+\varepsilon(z_0+\varepsilon z_1 + \dots))^2} - \varepsilon\alpha(\dot{z}_0 + \varepsilon\dot{z}_1 + \dots)^3$$

$$= -1 + 2\varepsilon(z_0 + \varepsilon z_1 + \dots) + \mathcal{O}(\varepsilon^2) - \varepsilon\alpha(\dot{z}_0 + \varepsilon\dot{z}_1 + \dots)^3$$

$\mathcal{O}(1)$: $\ddot{z}_0 = -1$

$\dot{z}_0 = -\tau + a \qquad \dot{z}_0(0) = 1 = a$

$z_0 = -\frac{1}{2}\tau + \tau + \cancel{b} \qquad z_0(0) = 0 = b$

$\mathcal{O}(\varepsilon)$: $\ddot{z}_1 = 2z_0 - \alpha\dot{z}_0^3 = 2(-\frac{1}{2}\tau^2 + \tau) - \alpha(-\tau+1)^3$

$\ddot{z}_1 = -\tau^2 + \tau + \alpha(\tau-1)^3$

$\dot{z}_1 = -\frac{1}{3}\tau^3 + \frac{1}{2}\tau + \frac{\alpha}{4}(\tau-1)^4 + c$

$\dot{z}_1(0) = 0 = \frac{\alpha}{4} + c \Rightarrow c = -\frac{\alpha}{4}$

$$\Rightarrow \dot{z} \sim -\tau + 1 + \varepsilon\left(-\frac{1}{3}\tau^3 + \frac{1}{2}\tau + \frac{\alpha}{4}(\tau-1)^4 - \frac{\alpha}{4}\right) \qquad (2)$$

We know $\dot{z}(\tau_a) = 0$ so consider:

$$\tau_a = \tau_0 + \varepsilon\tau_1 + \dots$$

and plug into (2):

$$\Rightarrow -(\tau_0 + \varepsilon\tau_1 + \dots) + 1 + \varepsilon\Big(-\tfrac{1}{3}(\tau_0 + \varepsilon\tau_1 + \dots)^3 + (\tau_0 + \varepsilon\tau_1 + \dots)^2$$

$$- \frac{\alpha}{4}(\tau_0 + \varepsilon\tau_1 + \dots) - \frac{\alpha}{4}\Big) = 0$$

$$O(1): \; -\tau_0 + 1 = 0$$

$$\tau_0 = 1$$

$$O(\varepsilon): \; -\tau_1 - \tfrac{1}{3}\tau_0^3 + \tau_0^2 - \frac{\alpha}{4}(\tau_0) - \frac{\alpha}{4} = 0$$

$$-\tau_1 - \tfrac{1}{3} + 1 - \frac{\alpha}{2} = 0$$

$$\tau_1 = \frac{2}{3} - \frac{\alpha}{2}$$

$$\boxed{\tau_a \sim 1 + \varepsilon\left(\frac{2}{3} - \frac{\alpha}{2}\right)}$$

Problem 2

a) $$\begin{cases} u_t = u_{xx} \\ u(x,0) = u(x,L) = 0 \\ u(0,t) = \sin(\omega t) \end{cases}$$

$$\hat{u}_t = \int_0^\infty e^{-st} \frac{\partial}{\partial t} u(x,t)\, dt = \cancel{e^{-st} u(x,t)\Big|_0^\infty}^{\,0} + s\int_0^\infty e^{-st} u(x,t)\, dt = s\hat{u}$$

$$\widehat{u_{xx}} = \hat{u}_{xx}$$

$$\Rightarrow s\hat{u} = \hat{u}_{xx}$$

$$\hat{u} = A(s) e^{-x\sqrt{s}} + B(s) e^{x\sqrt{s}}$$

b) $\hat{u}$ is analytic everywhere except at $s=0$ which is a b.p. Now we should pick a branch of $\sqrt{\ }$, I will choose the + branch. If $B(s) \neq 0$, then as $L \to \infty$ $u(L,t) \neq 0$ which contradicts our IC's. Thus we should pick $B(s) = 0$ to ensure a physically relevant soln. Hence

$$\hat{u}(x,s) = A(s) e^{-x\sqrt{s}}$$

$$u(x,t) = \frac{1}{2\pi i} \int_0^\infty A(s) e^{-x\sqrt{s}} e^{st}\, ds$$

$$A(s) = \hat{u}(0,s) = \int_0^\infty \sin(\omega t) e^{-st}\, dt = -\tfrac{1}{\omega}\cos(\omega t) e^{-st}\Big|_0^\infty - \tfrac{s}{\omega}\int_0^\infty \cos(\omega t) e^{-st}\, dt$$

$$u = e^{-st} \qquad v = -\tfrac{1}{\omega}\cos(\omega t)$$
$$du = -se^{-st}dt \qquad dv = \sin(\omega t)\, dt$$

$$= 1 - \frac{s}{\omega}\left(\cancel{\tfrac{1}{\omega}\sin(\omega t) e^{-st}\Big|_0^\infty}^{\,0} + \frac{s}{\omega}\int_0^\infty \sin(\omega t) e^{-st}\, dt \right)$$

$$= 1 - \left(\frac{s}{\omega}\right)^2 \int_0^\infty \sin(\omega t) e^{-st}\, dt$$

$$\left(1 + \frac{s^2}{\omega^2}\right)\int_0^\infty \sin(\omega t) e^{-st}\, dt = 1$$

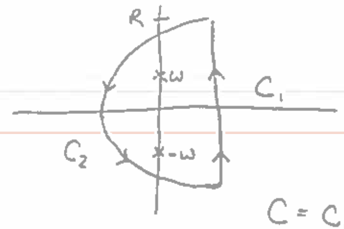
$$A(s) = \hat{u}(0,s) = \int_0^\infty \sin(\omega t) e^{-st}\, dt = \frac{\omega^2}{\omega^2 + s^2}$$

$$u(x,t)=\frac{\omega^2}{2\pi i}\int_{A-i\infty}^{A+i\infty}\frac{e^{-x\sqrt{s}}}{\omega^2+s^2}\,e^{st}\,ds=\frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty}\underbrace{e^{st}\,\hat{u}(x,s)}_{F(s,x)}\,ds$$

$$=\frac{\omega^2}{2\pi i}\int_{A-i\infty}^{A+i\infty}\frac{e^{st-x\sqrt{s}}}{s^2+\omega^2}\,ds$$

c) By Dominated Conv Thm

$$\lim_{x\to 0^+}u(x,t)=\frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty}\lim_{x\to 0^+}F(x,s)\,ds=\frac{\omega^2}{2\pi i}\int_{A-i\infty}^{A+i\infty}\frac{e^{st}}{s^2+\omega^2}\,ds$$



$$U(0,t)=\frac{\omega^2}{2\pi i}\int_C\frac{e^{st}}{s^2+\omega^2}\,ds=2\pi i\left(\operatorname{Res}\left(\frac{e^{st}}{s^2+\omega^2},\,s=\omega i\right)+\operatorname{Res}\left(\frac{e^{st}}{s^2+\omega^2},\,s=-\omega i\right)\right)$$

$C=C_1\cup C_2$

C_2: $s=\rho e^{i\theta}$ $\quad\theta\in(\alpha,2\pi-\alpha)\Rightarrow ds=i\rho e^{i\theta}d\theta$

$$\left|\int_\alpha^{2\pi-\alpha}i\rho e^{i\theta}\frac{e^{\rho e^{i\theta}t}}{\rho^2e^{2i\theta}+\omega^2}\,d\theta\right|=\left|\int_\alpha^{\pi-\alpha}i\rho e^{i\theta}\frac{e^{\rho e^{i\theta}t}}{\rho^2e^{2i\theta}+\omega^2}\,d\theta+\int_{\pi-\alpha}^{\pi+\alpha}i\rho e^{i\theta}\frac{e^{\rho e^{i\theta}t}}{\rho^2e^{2i\theta}+\omega^2}\,d\theta\right.$$

$$\left.+\int_{\pi+\alpha}^{2\pi-\alpha}i\rho e^{i\theta}\frac{e^{\rho e^{i\theta}t}}{\rho^2e^{2i\theta}+\omega^2}\,d\theta\right|$$

$$\le\int_\alpha^{\pi-\alpha}\cancelto{0}{\frac{\rho}{\rho^2-1}}e^{At}\,d\theta$$

$$+\int_{\pi-\alpha}^{\pi+\alpha}\cancelto{0}{\frac{\rho}{\rho^2-1}}e^{At}\,d\theta$$

$$+\int_{\pi+\alpha}^{2\pi-\alpha}\cancelto{0}{\frac{\rho}{\rho^2-1}}e^{At}\,d\theta\qquad\text{as }\rho\to\infty$$

C: $$\frac{\omega^2}{2\pi i}\int_C\frac{e^{st}}{s^2+\omega^2}\,ds=\frac{2\pi i\,\omega^2}{2\pi i}\left(\operatorname{Res}\left(\frac{e^{st}}{(s+i\omega)(s-i\omega)},\,i\omega\right)+\operatorname{Res}\left(\frac{e^{st}}{(s+i\omega)(s-i\omega)},\,-\omega i\right)\right)$$

$$=\omega^2\left(\frac{e^{i\omega t}}{2i\omega}+\frac{e^{-i\omega t}}{-2i\omega}\right)=\omega\left(\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right)=\omega\sin(\omega t)$$

By Cauchy's Thm

$$\frac{\omega^2}{2\pi i}\int_{C_1}\frac{e^{st}}{s^2+\omega^2}\,ds=\frac{\omega^2}{2\pi i}\int_C\frac{e^{st}}{s^2+\omega^2}\,ds-\frac{\omega^2}{2\pi i}\int_{C_2}\frac{e^{st}}{s^2+\omega^2}\,ds\longrightarrow\omega\sin\omega t+\overset{?}{0}=\omega\sin(\omega t)$$

as $R\to\infty$

Problem 3

$$I(a,b) = \int_0^b \frac{dx}{a + i\cos x}$$

a) $$I(a,2\pi) = \int_0^{2\pi} \frac{dx}{a+i\cos x}$$

$$= \int_0^{2\pi} \frac{a - i\cos x}{a^2 + \cos^2 x}\, dx$$

$$= \underbrace{\int_0^{2\pi} \frac{a}{a^2+\cos^2 x}\, dx}_{\in \mathbb{R}} - i\underbrace{\int_0^{2\pi} \frac{\cos x}{a^2+\cos^2 x}\, dx}_{I_1 = 0}$$ then ✓

$$I_1 = \int_0^{\pi} \frac{\cos x}{a^2+\cos^2 x}\, dx + \int_{\pi}^{2\pi} \frac{\cos x}{a^2+\cos^2 x}\, dx$$

Change of var: $\tilde{x} = x - \pi$

$\Rightarrow \cos(x) = \cos(\tilde{x}+\pi) = -\cos(\tilde{x})$

$$= \int_0^{\pi} \frac{\cos x}{a^2+\cos^2 x}\, dx - \int_0^{\pi} \frac{\cos x}{a^2+\cos^2 x}\, dx = 0$$ ✓

b) Let $z = e^{ix} \Rightarrow dz = ie^{ix}\, dx$

$$-\frac{i}{z}\, dz = dx$$

$$I(a,b) = \int^b \frac{dx}{a+i\cos x}$$

$$= \int_1^{e^{ib}} \frac{-i/z}{a + \frac{z+\frac{1}{z}}{2} i}\, dz$$

$$= \int_1^{e^{ib}} \frac{2}{-2azi + z^2 + 1}\, dz$$

$$= \int_1^{e^{ib}} \frac{-2}{z^2 - 2azi + 1}\, dz$$

$$z = \frac{2ai \pm \sqrt{-4a^2-4}}{2}$$

$$= i(a \pm \sqrt{a^2+1})$$

$$= \int_1^{e^{ib}} \frac{-2}{\underbrace{(z + i(a+\sqrt{a^2+1}))}\underbrace{(z+i(a-\sqrt{a^2+1}))}}\, dz$$

$$\frac{-2}{(z-z_+)(z-z_-)} = \frac{A}{z-z_+} + \frac{B}{z-z_-} \Rightarrow -2 = A(z-z_-) + B(z-z_+)$$

$$\begin{cases} Az_- + Bz_+ = 2 \\ A + B = 0 \Rightarrow A = -B \end{cases}$$

$$-Bz_- + Bz_+ = -2$$

$$B(2\sqrt{a^2+1}) = -2$$

$$B = \frac{-1}{\sqrt{a^2+1}}$$

$$A = \frac{1}{\sqrt{a^2+1}}$$

$$I(a,b) = \frac{1}{\sqrt{a^2+1}}\left(\int_1^{e^{ib}} \frac{1}{z-z_+}\,dz - \int_1^{e^{ib}} \frac{1}{z-z_1}\,dz\right)$$

If $b = 2k\pi$, $k \in \mathbb{Z}$, we can use Cauchy's Residue Thm.

If $b \in \mathbb{C}$ and e^{ib} doesn't land on a pole pt, then we have a region where our integrands are analytic and we just use the end pts. If e^{ib} does land on a pole point I'd have to think more...

c) If the contour is closed we need to also consider (a).

If the contour is not closed we can say:

$$I(a,b) = i\left[\ln(e^{ib} - z_+) - \ln(1-z_+) - \ln(e^{ib} - z_-) + \ln(1-z_-)\right] \; *$$

$$= u + iv \quad **$$

Format $*$ to match $**$ and set $v=0$ to find conditions on b.

Problem 4

Jan 2017

$$\begin{cases} u_t + \frac{1}{\varepsilon} v\left(\frac{x}{\varepsilon}\right) \partial_y u = \partial_x (K \partial_x u) + \partial_y (K \partial_y u) \\ u(x,y,0) = u_0(x,y) \end{cases}$$

$$K(x,\varepsilon) = A + G\left(\frac{x}{\varepsilon}\right)$$

Let $z = \frac{x}{\varepsilon} \Rightarrow \partial_x \longmapsto \partial_x + \frac{1}{\varepsilon}\partial_z$

Ansantz: $u = \bar{u} + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots$

$$\Rightarrow u_t + \frac{1}{\varepsilon} v(z) \partial_y u = (\partial_x + \frac{1}{\varepsilon}\partial_z)\left(K(\partial_x + \frac{1}{\varepsilon}\partial_z) u\right) + \partial_y (K \partial_y u)$$

$\mathcal{O}(1/\varepsilon^2)$: $\quad 0 = \partial_z (K \partial_z \bar{u})$

$$0 = \underbrace{(\partial_z K)}_{\text{nonzero}}(\partial_z \bar{u}) + \underbrace{K}_{\text{Nonzero}} \partial_z^2 \bar{u} \quad \Rightarrow \bar{u}(x,y,z,t) = \bar{u}(x,y,t)$$

$\mathcal{O}(1/\varepsilon)$: $\quad v(z)\partial_y \bar{u} = \partial_x (K \cancelto{0}{\partial_z \bar{u}}) + \partial_z (K \partial_x \bar{u}) + \partial_z (K \partial_z u_1)$

$$v(z)\partial_y \bar{u} = \partial_z \Big(K(\partial_z u_1 + \partial_x \bar{u})\Big)$$

$$\cancelto{0}{\langle v(z) \partial_y \bar{u} \rangle_z} = \langle \partial_z (K(\partial_z u_1 + \partial_x \bar{u})) \rangle_z$$

$\because$ v is periodic and $\bar{u}$ is indep of z

$$K(\partial_z u_1 + \partial_x \bar{u}) = A(x,y,t)$$

$$\partial_z u_1 + \partial_x \bar{u} = \frac{A(x,y,t)}{K(z)}$$

$$u_1 \text{ indep } z \Leftarrow \underbrace{\cancelto{0}{\langle \partial_z u_1 \rangle_z^h}}_{FA} + \langle \partial_x \bar{u} \rangle_z^h = A\left(\langle K \rangle_z^h\right)^{-1}$$

$$A = \langle K \rangle_z^h \partial_x \bar{u}$$

$$\boxed{\text{Note: } \partial_z u_1 + \partial_x \bar{u} = \frac{\langle K \rangle_z^h}{K} \partial \bar{u}}$$

$$O(1): \quad \underbrace{\bar{u}_t}_{\text{Want}} + v(z)\,\partial_y u_1 = \overset{*}{\partial_x (K \partial_z u_1)} + \partial_z (K \partial_x u_1) + \partial_z (K \partial_z u_2)$$

$$+ \underset{*}{\partial_x (K \partial_x \bar{u})} + \partial_y (K \partial_y \bar{u})$$

(Goal: Heat Eqn)

$$* \quad \partial_x (K \partial_z u_1 + K \partial_x \bar{u}) = \partial_x \left(\langle K \rangle_z^h \partial_x \bar{u} \right) = \langle K \rangle_z^h \partial_x^2 \bar{u}$$

$$\langle \bar{u}_t \rangle_z + \underbrace{\cancel{\langle v(z) \partial_y u_1 \rangle_z}}_{\because \text{ periodic}} \to 0 = \langle K \rangle_z^h \langle \partial_x^2 \bar{u} \rangle_z + \underbrace{\cancel{\langle \partial_z (K \partial_x u_1) \rangle_z}}_{\because \text{ FA}} \to 0$$

$$+ \underbrace{\cancel{\langle \partial_z (K \partial_z u_2) \rangle_z}}_{\because \text{ FA}} \to 0 + \langle \partial_y (K \partial_y \bar{u}) \rangle_z$$

(FA = anything w/ fast scale deriv goes to zero)

$$\Rightarrow \bar{u}_t = \underbrace{\langle K \rangle_z^h}_{\alpha} \bar{u}_{xx} + \underbrace{\langle K \rangle_z}_{\beta} \bar{u}_{yy}$$

$$\therefore \quad \boxed{\bar{u}(x,y,t) = \mathcal{F}^{-1} \left\{ c\, e^{(\alpha \chi^2 + \beta \gamma^2) t} \right\}}$$

where: $\mathcal{F}(\bar{u}_{xx}) = \chi^2$

and $\mathcal{F}(\bar{u}_{yy}) = \gamma^2$

and $\mathcal{F}(u_0(x,y)) = c$

Problem 5 Jan 2017

$$I(t,m) = \frac{\omega}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{t(s-m\sqrt{s})}}{\omega^2+s^2}\,ds$$

a) Let $g(s) = \frac{1}{\omega^2+s^2}$ and $h(s) = s - m\sqrt{s}$

$$\Rightarrow h'(s) = 1 - \frac{m}{2\sqrt{s}} \qquad \Rightarrow s^* = \frac{m^2}{4}$$

$$h''(s) = \frac{m}{4 s^{3/2}}$$

$$h(s^*) = \frac{m^2}{4} - m\left(\frac{m}{2}\right) = -\frac{m^2}{4}$$

$$h''(s^*) = \frac{8m}{4m^{3/2}} = \frac{1}{2\sqrt{m}} = \rho e^{i0} \Rightarrow \alpha = 0$$ where we pick the $+\sqrt{\ }$ branch

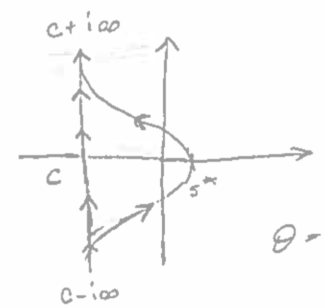
$$v = -\tfrac{1}{2}h''(s^*)(s-s^*)^2 = \rho e^{i0} \qquad \text{or} \qquad \theta = -\frac{\cancel{\alpha}\,0}{n} + (2m+1)\frac{\pi}{n}$$

$$n=2 \qquad m = 0,1$$

$$e^{i\pi}\frac{1}{4\sqrt{m}}\left(s-\frac{m^2}{4}\right)^2 = \rho e^{i0} > 0 \qquad \theta_0 = \frac{\pi}{2} \qquad \theta_1 = \frac{3\pi}{n}$$

$$\left(s-\frac{m^2}{4}\right)^2 = \rho e^{i\pi}$$

$$s - \frac{m^2}{4} = \rho e^{i\pi/2}$$



$\theta = \frac{\pi}{2}$

$$I(t,m) \sim \frac{\omega}{2\pi i}\cdot e^{th(s^*)}\sqrt{\frac{2\pi}{t|h''(s^*)|}}\; g(s^*)\,e^{i\theta}$$

$$= \frac{\omega}{2\pi i}\, e^{-tm^2/4}\sqrt{\frac{4\pi\sqrt{m}}{t}}\;\frac{1}{\omega^2+\frac{m^2}{4}}\; i$$

$$= e^{\frac{-tm^2}{4}}\;\frac{m^{1/4}}{\sqrt{\pi t}}\;\frac{4\omega}{4\omega^2+m^2}$$

$$\boxed{I(t,m) = \frac{4\omega m^{1/4}}{(4\omega^2+m^2)(\pi t)^{1/2}}\, e^{-tm^2/4}}$$

Problem 6

Jan 2017

a)

$$\varepsilon z^5 + (z+1)^2 = \varepsilon \qquad (1)$$

Consider the unpreturbed problem:

$$(z+1)^2 = 0$$
$$\Rightarrow z = -1$$

Thus:

$$z_0 = \sum_{n=0}^{\infty} a_n \varepsilon^{n/2} \qquad \text{where } a_0 = -1$$

Plug z_0 into (1):

$$\varepsilon(-1 + a_1\varepsilon^{1/2} + \dots)^5 + (\cancel{-1} + a_1\varepsilon^{1/2} + \dots + \cancel{1})^2 = \varepsilon$$

$\Theta(\varepsilon^{1/2})$: $0 = 0$

$\Theta(\varepsilon)$: $-1 + a_1^2 = 1$

$$a_1 = \pm\sqrt{2}$$

$$\Rightarrow \boxed{\begin{aligned} z_1 &\sim -1 + \sqrt{2}\,\varepsilon^{1/2} \\ z_2 &\sim -1 - \sqrt{2}\,\varepsilon^{1/2} \end{aligned}}$$

Let $x = \varepsilon^{\alpha} z \Rightarrow z = \varepsilon^{-\alpha} x$ and plug into (1):

$$\varepsilon^{1-5\alpha} x^5 + \varepsilon^{-2\alpha} x^2 + 2\varepsilon^{-\alpha} x + 1 = \varepsilon$$

$$1 - 5\alpha = -2\alpha$$
$$\alpha = 1/3$$

$$\Rightarrow \varepsilon^{-2/3} x^5 + \varepsilon^{-2/3} x^2 + 2\varepsilon^{-1/3} x + 1 = \varepsilon$$

$$x^5 + x^2 + 2\varepsilon^{1/3} x + \varepsilon^{2/3} = \varepsilon^{5/3} \qquad (2)$$

Consider the unpreturbed version of (2):

$$x^5 + x^2 = 0$$

$$x^2(x^3+1)=0 \qquad w = -e^{i\frac{2\pi}{3}}$$

$$x = 0, 0, w, w^2, -1$$

correspond to z_1, z_2

$$x_3 = w + \sum_{n=1}^{\infty} b_n \varepsilon^{n/3}$$

$$x_4 = w^2 + \sum_{n=1}^{\infty} c_n \varepsilon^{n/3}$$

$$x_5 = -1 + \sum_{n=1}^{\infty} d_n \varepsilon^{n/3}$$

Plug x_3 into (2):

$$(w + b_1\varepsilon^{1/3} + \dots)^5 + (w + b_1\varepsilon^{1/3} \dots)^2 + 2\varepsilon^{1/3}(w + b_1\varepsilon^{1/3} + \dots) + \varepsilon^{2/3} = \varepsilon^{5/3}$$

$O(1)$: $-w^2 + w^2 = 0$

Correction:

$O(\varepsilon^{1/3})$: $5w^{\cancel{4}\,3} b_1 + 2wb_1 + 2w = 0$

$$b_1 = \frac{-2}{5w^3+2} = \frac{2}{3}$$

Similarly: $c_1 = \dfrac{-2}{5w+2}$

$$d_1 = \frac{-2}{5w^2+2}$$

Recall: $z = \varepsilon^{-1/3} x$

$$\Rightarrow \boxed{\begin{aligned} z_3 &\sim \frac{w}{\varepsilon^{1/3}} + \frac{2}{3} \\ z_4 &\sim \frac{w^2}{\varepsilon^{1/3}} - \frac{2}{5w+2} \\ z_5 &\sim -\frac{1}{\varepsilon^{1/3}} - \frac{2}{5w^2+2} \end{aligned}}$$

Jan 2017

b) $x^2 e^{-x^2} = \varepsilon$

$$e^{-x^2} \sim 1 - x^2 + \frac{x^4}{2} + \ldots \sim 1 - x^2$$

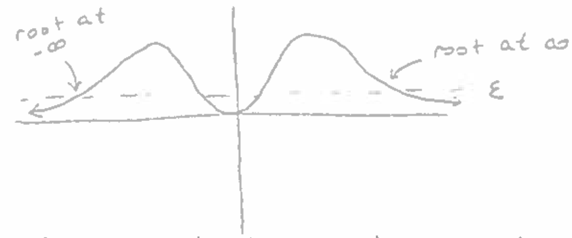
Taylor series

$$\Rightarrow x^2(1-x^2) = \varepsilon$$

$$x^4 - x^2 + \varepsilon = 0$$

$$x^2 = \frac{+1 \pm \sqrt{1-4\varepsilon}}{2}$$

$$x = \pm\sqrt{\frac{1 \pm \sqrt{1-4\varepsilon}}{2}}$$



So we only have two roots close to $x=0$. If we choose $\varepsilon = 0$:

$$x = \pm\sqrt{\frac{1+1}{2}} \quad \text{or} \quad x = \pm\sqrt{\frac{1-1}{2}}$$

makes no sense   makes sense

$$\Rightarrow \boxed{x_{1,2} = \pm\sqrt{\frac{1-\sqrt{1-4\varepsilon}}{2}}}$$

<u>Bonus:</u> $e^{-x^2} = \frac{\varepsilon}{x^2} \Rightarrow$ consider $e^{-x^2} = \varepsilon$

$$-x^2 \ln e = \ln \varepsilon$$

$$-x^2 = \ln \varepsilon$$

$$x^2 = \ln {}^1\!/\!_\varepsilon$$

$$x = \pm\sqrt{\ln(1/\varepsilon)}$$

$$x_0 = \sqrt{\ln(1/\varepsilon)}$$

But

$$x = \pm\sqrt{\ln\left(\frac{x^2}{\varepsilon}\right)}$$

$$\Rightarrow x_{n+1} = \sqrt{\ln\left(\frac{x_n^2}{\varepsilon}\right)} \Rightarrow x_1 = \sqrt{\ln\left(\frac{x_0^2}{\varepsilon}\right)} = \sqrt{2\ln x_0 - \ln \varepsilon}$$

$$= \sqrt{2\ln(\ln{}^1\!/\!_\varepsilon)^{1/2} - \ln\varepsilon}$$

$$= \sqrt{\ln(\ln(1/\varepsilon)) + \ln(1/\varepsilon)}$$

Problem 7

Jan 2017

$$T_t + \gamma x T_y = K(T_{xx} + T_{yy}) \qquad (1)$$

$$T(x,y,0) = T_0(\alpha x, \alpha y)$$

a) $\begin{cases} \alpha x = z \\ \alpha y = w \end{cases}$ where z and w are non-dimensional

$\Rightarrow \boxed{[\alpha] = \frac{1}{L}}$

$\Rightarrow z = \frac{x}{L} \qquad w = \frac{y}{L}$

Let $\tau = \frac{t}{T}$

$\Rightarrow \frac{\partial}{\partial y} = \frac{1}{L}\frac{\partial}{\partial w}$

$\frac{\partial}{\partial t} = \frac{1}{T}\frac{\partial}{\partial \tau}$

$(1) \Rightarrow \frac{1}{T} T_\tau + \gamma(zL)\frac{1}{L} T_w = \frac{K}{L^2}(T_{zz} + T_{ww})$

$T_\tau + \gamma T z T_w + \frac{KT}{L^2}(T_{zz} + T_{ww})$

$\bar{\gamma} = \gamma T = \text{non-dim}$ so $[\gamma] = \frac{1}{T}$

$\bar{K} = \frac{KT}{L^2} = \text{non-dim}$ so $[K] = \frac{L^2}{T}$

$$\therefore \; T_\tau + \bar{\gamma} z T_w = \bar{K}(T_{zz} + T_{ww}) \qquad (2)$$

is non-dim

b) $\widehat{x T_y} = \int_{\mathbb{R}^2} e^{-ikx} e^{-i\eta y} x\, T_y \, dx\, dy$

$$= -\frac{1}{i} \frac{\partial}{\partial k} \int_{\mathbb{R}^2} e^{-ikx} e^{-i\eta y} T_y \, dx\, dy$$

$$= i \frac{\partial}{\partial k} \int_{\mathbb{R}} e^{-ikx} \left(\int_{\mathbb{R}} e^{-i\eta y} T_y \, dy \right) dx$$

$$= i \frac{\partial}{\partial k} \int_{\mathbb{R}} e^{-ikx} \Big(\underbrace{e^{-i\eta y} T \Big|_{-\infty}^{\infty}}_{=0} + i\eta \int_{\mathbb{R}} e^{-i\eta y} T \, dy \Big) \, dx$$

$$= -i\eta \frac{\partial}{\partial k} \int_{\mathbb{R}^2} e^{-ikx} e^{-i\eta y} T \, dy\, dx$$

$$= i\eta \frac{\partial}{\partial k} \hat{T} \quad - i\eta \hat{T}_k$$

$$\widehat{T_{xx}} = (-ik)^2 \hat{T} = -k^2 \hat{T} \qquad \widehat{T_\tau} = \hat{T}_\tau$$

$$\widehat{T_{yy}} = (-i\eta)^2 \hat{T} = -\eta^2 \hat{T}$$

Let $\hat{T} = u$ (for ease of notation)

$$\mathcal{F}\{(2)\} \Rightarrow \quad u_\tau + \bar{\gamma}(-\eta u_k) = -\bar{k}(k^2+\eta^2) u$$

$$u_\tau - \bar{\gamma}\eta u_k = -\bar{k}(k^2+\eta^2) u \qquad (3)$$

<u>Method of Characteristics</u>

$$\begin{cases} z(s) = u(K(s), \eta(s), \Upsilon(s)) \\ K(0) = k \\ \eta(0) = \eta \\ \Upsilon(0) = \tau \end{cases}$$

$$\begin{cases} \dfrac{d\kappa}{ds} = -\bar{\gamma}\eta \\[2mm] \dfrac{d\eta}{ds} = 0 \\[2mm] \dfrac{d\tau}{ds} = 1 \end{cases} \Rightarrow \begin{cases} \kappa(s) = -\bar{\gamma}\eta s + \underset{\uparrow\, \kappa(0)}{\kappa} \\[2mm] \eta(s) = \eta \leftarrow \eta(0) \\[2mm] \tau(s) = s + \underset{\uparrow\, \tau(0)}{\tau} \end{cases}$$

$$z_s = u_\kappa \frac{d\kappa}{ds} + u_\eta \frac{d\eta}{ds} + u_\tau \frac{d\tau}{d\tau}$$

$$z_s = -\bar{k}\left(\kappa^2(s) + \eta^2(s)\right) z$$

$$\frac{dz}{z} = -\bar{k}\left((-\bar{\gamma}\eta s + \kappa)^2 + \eta^2\right) ds$$

$$\Rightarrow z(s) = z(0)\, e^{-\bar{k}\int_0^s \left[(-\bar{\gamma}\eta v + \kappa)^2 + \eta^2\right] dv}$$

so solve for $z(0)$

$$z(0) = u(\kappa(0), \eta(0), \tau(0)) = u(\kappa, \eta, \tau)$$

$$\Rightarrow u(\kappa, \eta, \tau) = z(s)\, e^{\bar{k}\left[\left.\frac{(-\bar{\gamma}\eta v + \kappa)^3}{-3\bar{\gamma}\eta}\right|_0^s + \eta^2 s\right]}$$

$$= z(s)\, e^{\bar{k}\left(\frac{\kappa^3 - (-\bar{\gamma}\eta s + \kappa)^3}{3\bar{\gamma}\eta} + \eta^2 s\right)}$$

Let $s = -\tau$

$$\Rightarrow z(t) = u(\bar{\gamma}\eta\tau + \kappa, \eta, 0)$$

$$= u_0(\bar{\gamma}\eta\tau + \kappa, \eta)$$

$$= \hat{T}_0(\bar{\gamma}\eta\tau + \kappa, \eta) = T_0$$

w/ help from $\mathcal{F}^{-1}$

$$\boxed{T(x,y,t) = \int_{\mathbb{R}^2} e^{i(\kappa x + \eta y)}\, \hat{T}_0\, e^{\bar{k}\left(\frac{\kappa^3 - (\bar{\gamma}\eta t + \kappa)^3}{3\bar{\gamma}\eta}\right)}\, e^{-\eta^2 t}\, d\kappa\, ds}$$

Problem 8

a) Sps $\{f_n\}$ is a seq of funcs defined on Ω.

<u>Ptws</u> We say $f_n \to f$ if $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ st $\forall n \geq N$ we get $|f_n(x_0) - f(x_0)| < \varepsilon$.

Only concerned with the fixed pt x_0 and N changes.

<u>Asymp</u>: For a fixed N, we say $f_N \to f$ if $\forall \varepsilon > 0 \; \exists \delta > 0$ st if $|x - x_0| < \delta$ we have $|f_N(x) - f(x)| < \varepsilon$.

Only concerned w/ nbhd of x_0, not x_0 itself and N is fixed.

Ex1: $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$

Ratio Test: $\displaystyle\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1} \xrightarrow[n \to \infty]{x \text{ fixed}} 0 \quad \therefore$ ptws conv

$\displaystyle\xrightarrow[x \to \infty]{n \text{ fixed}} \infty \quad \therefore$ not asymp conv

Ex2: $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{x^n}$

Ratio Test: $\displaystyle\frac{(n+1)!}{x^{n+1}} \cdot \frac{x^n}{n!} = \frac{n+1}{x} \xrightarrow[n \to \infty]{x \text{ fixed}} \infty \quad \therefore$ not ptws conv

$\displaystyle\xrightarrow[x \to \infty]{n \text{ fixed}} 0 \quad \therefore$ asymp conv

b) If f is uniformly continuous and integrable near x_0.

(These conditions might not be necessary but they should be sufficient.)