Ordinary Differential Equations: Toolbox

Math 383 - Gracie Conte

FIRST ORDER EQUATIONS

- **Separation of Variables**

  1. Isolate y's on once side and x's on the other
  2. Integrate and solve as needed

- **Integration Factor** (Linear)

  1. Given: $y' + P(x)y = Q(x)$
  2. Let $\mu = e^{\int^x P(t)\,dt}$
  3. Multiply by μ: $\mu y' + \mu P(x)y = \mu Q(x)$
  4. Undo product rule
  5. Integrate and solve as needed

- **Exact**

  1. Given: $M(x,y)\,dx + N(x,y)\,dy = 0$
  2. Check exactness: $M_y \stackrel{?}{=} N_x$
     - If equal then they are exact
       (a) $F(x,y) = \int F_x\,dx = \int M\,dx + g(y)$ $\big($or $F(x,y) = \int F_y\,dy = \int N\,dy + g(x)\big)$.
       (b) Find what $g(y)$ is by finding F_y $\big($or find what $g(x)$ is by finding $F_x\big)$.
       (c) Plug $g(y)$ (or $g(x)$) into $F(x,y)$.
       (d) Your solution is $F(x,y) = c$ where you plug in what you found for $F(x,y)$.
     - If not equal then not exact
       (a) Check if $\frac{M_y - N_x}{N}$ is a function of **just** x. If it isn't, stop.
       (b) Let $f(x) = \frac{M_y - N_x}{N}$
       (c) Let $\mu(x) = e^{\int^x f(t)\,dt}$
       (d) Multiply by $\mu(x)$: $\mu(x)M(x,y)\,dx + \mu(x)N(x,y)\,dy = 0$
       (e) Now $\bar{M} = \mu M$ and $\bar{N} = \mu N$
       (f) Check for exactness using $\bar{M}$ and $\bar{N}$. (If it isn't exact, something went wrong.)
       (g) Proceed like before in the exact case using $\bar{M}$ and $\bar{N}$ instead of M and N.

- **Substitution**
  - Homogeneous
    1. Let $u = \frac{y}{x}$ then $y = xu$
    2. Find $\frac{dy}{dx}$ in terms of u
    3. Solve in terms of u then use $u = \frac{y}{x}$ to recover y solution.

  - Bernoulli
    1. Given: $y' + P(x)y = Q(x)y^n$
    2. Let $u = y^{1-n}$ then after a couple steps you get $\frac{1}{1-n} y^n \frac{du}{dx} = \frac{dy}{dx}$
    3. Make substitutions then divide by y^n
    4. Solve in terms of u then recover y solution using the substitution $u = y1 - n$

  - Second Order Reducible
    * Case 1: No y terms
      1. Let $u = \frac{dy}{dx}$
      2. Then $\frac{du}{dx} = \frac{d^2y}{dx^2}$
      3. Solve in terms of u and x then substitute $u = y'$ back in.
      4. Solve new first order equation

    * Case 2: No x terms
      1. Let $u = \frac{dy}{dx}$
      2. Then $\frac{d^2y}{dx^2} = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy}u$
      3. Solve in terms of u and x then substitute $u = y'$ back in.
      4. Solve new first order equation

Nth ORDER EQUATIONS

- $a_n y^{(n)} + ... + a_1 y' + a_0 y = 0$ (Linear Homogeneous Constant Coefficient)
  - Solve characteristic equation: $a_n r^n + ... + a_1 r + a_0 = 0$
    * Real roots
      · Distinct: $r_0 \to e^{r_0 x}$
      · Repeated with multiplicity k: $r_0 \to e^{r_0 x}, xe^{r_0 x}, ..., x^{k-1} e^{r_0 x}$
    * Complex roots
      · Distinct: $t \pm is \to e^{tx}\cos(sx), e^{tx}\sin(sx)$
      · Repeated with multiplicity k: $t \pm is \to e^{tx}\cos(sx), xe^{tx}\cos(sx), ..., x^{k-1}e^{tx}\cos(sx)$
      $e^{tx}\sin(sx), xe^{tx}\sin(sx), ..., x^{k-1}e^{tx}\sin(sx)$

- $ay'' + by' + cy = f(x)$ (Second Order Linear Non-homogeneous Constant Coefficient)
  - Method of Undetermined Coefficients
    1. Find the homogeneous solution y_h
    2. Guess particular solution y_p has similar form to $f(x)$ then solve for coefficients. (Multiply by x if your guess has overlap with y_h.)
    3. $y = y_p + y_h$
  - Variation of Parameters
    1. Find the homogeneous solution y_h
    2. Particular solution $y_p = -u_1 y_1 + u_2 y_2$ where

$$u_1 = \int \frac{y_2 f(x)}{W(y_1, y_2)}\, dx \qquad u_2 = \int \frac{y_1 f(x)}{W(y_1, y_2)}\, dx$$

    3. $y = y_p + y_h$

SYSTEMS of EQUATIONS

- Find eigenvalues $\lambda_1, ..., \lambda_n$
- Find associated eigenvectors
- If λ_k is **real distinct**
  - General Solution: $x_k(t) = c_k e^{\lambda_k t} \vec{v}_k$
- If λ_k is **real repeated** (Multiplicity 2)
  - Easy Case: Two rows of zeros
    * Define two linearly independent vectors $\vec{w}_1$ and $\vec{w}_2$
    * General Solution: $x_k(t) = c_1 e^{\lambda_k t} \vec{w}_1 + c_2 e^{\lambda_k t} \vec{w}_2$
  - Hard Case: One row of zeros
    * Find the one vector that does work $\vec{w}_1$
    * Solve $(A - \lambda I)\vec{w}_2 = \vec{w}_1$ for $\vec{w}_2$
    * General Solution: $x_k = c_1 e^{\lambda_k t} \vec{w}_1 + c_2 (t e^{\lambda_k t} \vec{w}_1 + e^{\lambda_k t} \vec{w}_2)$
- If λ_k is **complex**
  - General Solution: $x_k(t) = c_k e^{(a_k + i b_k)tx} \vec{v}_k$
  - Convert using Euler's Formula: $e^{(a_k + i b_k)t} = e^{a_k t}(\cos(b_k t) + i \sin(b_k t)$
  - Multiply result from previous step into vector $\vec{v}_k$
  - Organize the resulting vector so each entry has the real portion first and imaginary portion second
  - Split this into two vectors so that it has the form $e^{a_k t}(\vec{w}_1 + i\vec{w}_2)$
  - New General Solution: $x_k(t) = c_1 e^{a_k t} \vec{w}_1 + c_2 e^{a_k t} \vec{w}_2$
- Full general solution is the sum of all the x_k's found