METHODS OF APPLIED MATHEMATICS COMPREHENSIVE EXAMINATION JANUARY 2014

Work on as many of the following problems as possible. Turn in *all* your work.

(1) Consider a projectile fired vertically from the surface of the Earth. Assume there is an inverse quadratic, attractive gravitational force field ($F = -\frac{GM_e m}{R^2}$, where R distance between the projectile and the Earth's center, M_e, m are the respective Earth and projectile mass, and G is the universal gravitational constant.)

(a) Write down the coupled dynamics for the projectile-Earth system.

(b) Assuming that the Earth mass is far greater than the projectile mass, decouple the dynamics.

(c) Measure the projectile height from the Earth surface by introducing R_e as the radius of the Earth. Then, identify standard kinematics by letting $g = -\frac{GM_e}{R_e^2} \approx -9.8m/s^2$. Identify characteristic length and time scales in terms of g and the initial velocity, v_0. Non-dimensionalize the problem, and identify the non-dimensional parameter and simplified non-linear ODE governing the evolution.

(d) Assuming that the projectile speed is sufficiently small, write down a two term perturbation expansion for the time to return to the Earth surface.

(2) Consider the following integral with real parameters (ϵ, a),

$$\int_0^\infty \frac{dx}{(x+\epsilon)^a(1+x)}$$

Hinch Analysis

Find the leading order asymptotic expansion as $\epsilon \to 0^+$ as the parameter a varies in $(0, \infty)$. Discuss all possible cases.

(3) By applying iterated homogenization, average the following equation to derive a leading order effective equation governing the evolution as $\epsilon \to 0$, assuming the functions $K(x), M(x)$ and $H(x)$ are periodic sharing the same period and positive:

$$u_t = \left[\left(K(x/\epsilon^2) + M(x/\epsilon^3)\right) u_x\right]_x + H(x/\epsilon^p)u$$
$$u(x, t=0) = u_0(x);$$

distinguish the different behaviors as the parameter p takes the integer values $p = 1, 2, 3, 4$.

(4) Consider the following "quarter-plane" boundary value problem for the equation

$$c_t = \kappa\, c_{xxx}, \qquad c(x,0) = 0, \qquad c(0,t) = t$$

where κ is a fixed nonzero real parameter.

(a) Solve for all time $t > 0$ in the half line $x \geq 0$.

(b) Discuss the differences, if any, between the case $\kappa > 0$ and $\kappa < 0$.
(c) Discuss the limit $x \to 0^+$ (verify the boundary condition is satisfied).

(5) Find a two term asymptotic expansion for all the roots of the following as $\epsilon \to 0$

$$(x-1)^2 + \epsilon x^{\frac{1}{3}} = 0 .$$

(6) Consider the function $F(s;x)$ dependent on the real parameter $x > 0$,

$$F(s;x) = \frac{1}{s^2} \exp(-x s^{\frac{1}{3}}) .$$

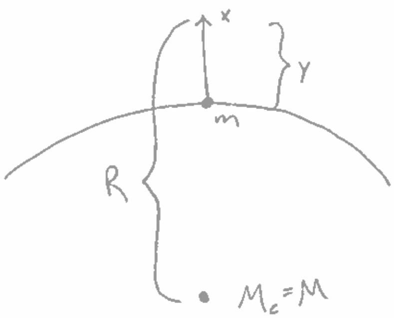
(a) Classify its singularities and define domains where it is single valued.
(b) Choose the domains in such a way that the integral depending on the parameter $c > 0$,

$$I(x,t) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{st} F(s;x)\, ds ,$$

is well defined, and in particular discuss its convergence.
(c) Evaluate the integral along appropriate contours in the s-complex plane and discuss the first term of the asymptotic expansion of $I(x,t)$ as $t \to +\infty$.

(7) (a) Explain the difference between pointwise convergence and asymptotic convergence. Illustrate with the particular example of power series.
(b) Are asymptotic series unique? Explain.
(c) Is the asymptotic relation associative? Discuss.

a)



$$\begin{cases} m\ddot{x} = -\dfrac{GMm}{R^2} \\ M\ddot{x}_e = -\dfrac{GMm}{R^2} \end{cases} \qquad (1)$$

b) Since $M \gg m \Rightarrow V_e \ll V \Rightarrow \dot{x}_e \approx \text{const} \Rightarrow \ddot{x}_e = 0$

Then the prob reduces to:

$$\begin{cases} \ddot{x} = -\dfrac{GM}{R^2} \\ \dot{x}(0) = v_0 > 0 \\ x(0) = R_e = \text{radius of earth} \end{cases} \qquad (2)$$

c) Let $\begin{cases} y = x - R_e \\ x_e = 0 \end{cases} \Rightarrow x = y + R_e$

$$\Rightarrow \ddot{y} = -\frac{GM}{|y+R_e|^2} = -\frac{GM}{R_e^2|\frac{y}{R_e}+1|^2}$$

$$\ddot{y} = -\frac{GM}{R_e^2} \cdot \frac{1}{|1+y/R_e|^2} = -g\frac{1}{(1+y/R_e)^2} \qquad (3)$$

where $y(0) = x(0) - R_e = 0$

$\dot{y}(0) = \dot{x}(0) = v_0 > 0$

Consider $\frac{y}{R_e} \approx 0$; $y \ll R_e$

$$\Rightarrow \ddot{y} = -g$$

$$\dot{y} = -gt + b \qquad \dot{y}(0) = b = V_0$$

$$y = -\tfrac{1}{2}gt^2 + v_0 t + c \qquad y(0) = c = 0$$

$$\dot{y}(t_a) = 0 \Rightarrow -t_a g + V_0 = 0 \Rightarrow t_a = \frac{v_0}{g}$$

$$\Rightarrow \ddot{y}(t_a) = -\tfrac{1}{2}g t_a^2 + v_0 t_a = -\frac{v_0^2 g}{2g^2} + \frac{v_0^2}{g} = \frac{v_0^2}{2g}$$

Non-Dimentionalization

Choose $T = \frac{v_0}{g}$ and $L = \frac{v_0^2}{g}$ and then let

$t = \tau T$ and $y = zL$.

dimentionless

$$\Rightarrow \dot{y} = L\frac{dz}{dt} = L\frac{dz}{d\tau}\frac{d\tau}{dt} = \frac{L}{T}\frac{dz}{d\tau} = \frac{L}{T}\dot{z}$$

$$\ddot{y} = \frac{d}{dt}\left(\frac{L}{T}\frac{dz}{d\tau}\right) = \frac{L}{T}\frac{d^2z}{d\tau^2}\frac{d\tau}{dt} = \frac{L}{T^2}\ddot{z}$$

Sub into (3)

$$\frac{L}{T^2}\ddot{z} = -g\frac{1}{\left(1+\frac{zL}{R_e}\right)^2} \Rightarrow \cancel{g}\ddot{z} = -\cancel{g}\frac{1}{\left(1+z\frac{L}{R_e}\right)^2}$$

$$\boxed{\ddot{z} = \frac{1}{(1+\varepsilon z)^2}} \quad \text{where } \varepsilon = \frac{L}{R_e} \text{ and} \qquad (4)$$

$$\cancel{v_0} = \dot{y}(0) = L\frac{dz}{dt}\Big|_{t=0} = \frac{L}{T}\frac{dz}{d\tau}\Big|_{\tau=0} = \cancel{v_0}\,\dot{z}(0) \Rightarrow \dot{z}(0) = 1$$

$$0 = y(0) = z(0)\cdot L \qquad \Rightarrow z(0) = 0$$

d) For sufficiently small v_0, $L = \frac{v_0^2}{g} << R_e$ thus ε is also small. So consider;

$$z \sim z_0 + \varepsilon z_1 + \ldots$$

Plug this into (4):

$$(\ddot{z}_0 + \varepsilon \ddot{z}_1) = -\frac{1}{(1+\varepsilon(z_0+\varepsilon z_1))^2} = -1 + 2\varepsilon(z_0 + \varepsilon z_1) + \ldots$$

Taylor series

$O(1)$:
$$\ddot{z}_0 = -1$$
$$\dot{z}_0 = -\tau + b \qquad \dot{z}_0(0) = b = 1$$
$$z_0 = -\frac{1}{2}\tau^2 + \tau + c \qquad z_0(0) = c = 0$$

$O(\varepsilon)$:
$$\ddot{z}_1 = 2z_0 = -\tau^2 + 2\tau$$
$$\dot{z}_1 = -\frac{1}{3}\tau^3 + \tau^2 + d \qquad \dot{z}_1(0) = d = 0$$
$$z_1 = -\frac{1}{12}\tau^4 + \frac{1}{3}\tau^3 + f \qquad z_1(0) = f = 0$$

$$z \sim -\frac{1}{2}\tau^2 + \tau + \varepsilon\left(-\frac{1}{12}\tau^4 + \frac{1}{3}\tau^3\right) \qquad (5)$$

We know $z(\tau_R) = z(0) = 0$. So consider $\tau_R = \tau_0 + \varepsilon \tau_1 + \ldots$ and plug this into $(5) = 0$.

$$-\frac{1}{2}(\tau_0+\varepsilon\tau_1)^2+(\tau_0+\varepsilon\tau_1)+\varepsilon\left(-\frac{1}{12}(\tau_0+\varepsilon\tau_1)^4+\frac{1}{3}(\tau_0+\varepsilon\tau_1)^3\right)=0$$

$O(1)$: $-\frac{1}{2}\tau_0^2+\tau_0=0$

$$\tau_0(1-\tau_0/2)=0$$

$$\tau_0=\underset{\text{start}}{0},\ \underset{\text{return}}{2}$$

$O(\varepsilon)$: $-\tau_1\tau_0+\tau_1-\frac{1}{12}\tau_0^4+\frac{1}{3}\tau_0^3=0$

$$-\tau_1=\frac{1}{12}(2)^4-\frac{1}{3}(2)^3=\frac{4}{3}-\frac{8}{3}=-4/3$$

$$\tau_1=4/3$$

$$\boxed{\tau_R=2+\frac{4}{3}\varepsilon}$$

Problem 4

$$(x-1)^2 + \varepsilon x^{1/3} = 0 \tag{1}$$

$\varepsilon = 0 \Rightarrow \quad (x-1)^2 = 0$

$x = 1, 1$

$x_{1,2} \sim 1 + a\varepsilon^{\alpha}$

Sub into (1):

$$(1 + a\varepsilon^{\alpha} - 1)^2 + \varepsilon(1 + a\varepsilon^{\alpha})^{1/3} = 0$$

$$a^2\varepsilon^{2\alpha} = -\varepsilon(1 + a\varepsilon^{\alpha})^{1/3} \tag{2}$$

$$a^6\varepsilon^{6\alpha} = -\varepsilon^3(1 + a\varepsilon^{\alpha})$$

$$a^6\varepsilon^{6\alpha} - a\varepsilon^{\alpha+3} + \varepsilon^3 = 0$$

$6\alpha = 3$

$\alpha = 1/2 \Rightarrow$ consistent

$$a^6\varepsilon^3 - a\varepsilon^{7/2} + \varepsilon^3 = 0$$

$O(\varepsilon^3)$: $a^6 + 1 = 0$

$$a^6 = e^{(2k+1)i\pi}$$

$$a = e^{(2k+1)\pi i/6} = \omega \quad k = 0, 1, \ldots, 5$$

However there should only be two roots. We got 6 b/c we have 2 for each branch of the cube root.

Plug a and α into (2):

$$e^{\frac{(2k+1)\pi i}{6}\cdot 2}\varepsilon = -\varepsilon\left(1 + e^{\frac{(2k+1)\pi i}{6}}\varepsilon^{1/2}\right)^{1/3}$$

$$e^{\frac{(2k+1)\pi i}{3}} = e^{\pi i}\left(1 + e^{\frac{(2k+1)\pi i}{6}}\varepsilon^{1/2}\right)^{1/3}$$

$\vdots$

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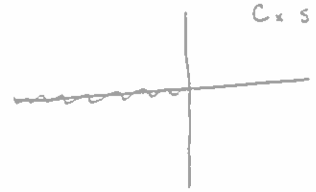
Problem 6

$$F(s,x) = \frac{e^{-xs^{1/3}}}{s^2}$$

a) $f(s) = s^{1/3}$ $\quad s=0$ b.p.

$g(s) = \frac{1}{s^2}$ $\quad s=0$ double pole

Choose branch cut:



$\leftarrow$ domain where $F(s,x)$ is sing. val.

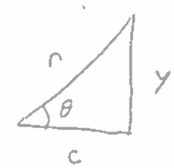
b) $$|I(x,t)| \le \left| \int_{c-i\infty}^{c+i\infty} \frac{e^{st-xs^{1/3}}}{s^2}\, ds \right|$$

$s = c+iy \Rightarrow ds = i\,dy$

or

$s = re^{i\theta} \quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\le \int_{-\infty}^{\infty} \frac{e^{ct-xr^{1/3}\cos\frac{\theta}{3}}}{r^2}\, dy$$



$r = \sqrt{c^2+y^2}$

$$\le \int_{-\infty}^{\infty} \frac{e^{ct-x(c^2+y^2)^{1/3}\cos\frac{\pi}{6}}}{c^2+y^2}\, dy$$

$$\le \int_{-\infty}^{\infty} \frac{e^{ct-x\overbrace{(c^2+y^2)^{1/6}\cos\frac{\pi}{6}}^{>0}}}{c^2}\, dy$$

$< \infty$ if $x > 0$

The domain where $I(x,t)$ is well-defined and convergent is: $\{(x,t) \mid x>0,\ t\in\mathbb{R}\}$

c) $$I(x,t) = \int_{c-i\infty}^{c+i\infty} \frac{e^{st - xs^{1/3}}}{s^2} \sim \; ? \quad \text{as } t \to \infty$$

$$= \underbrace{-\left.\frac{e^{st - xs^{1/3}}}{s}\right|_{c-i\infty}^{c+i\infty}}_{\text{Part 1}} + \underbrace{\int_{c-i\infty}^{c+i\infty} \overbrace{\frac{e^{st - xs^{1/3}}}{s}}^{f(x,s,t)} \left(t - \frac{x}{s^{2/3}}\right) ds}_{\text{Part 2}}$$

Part 1: $s = c + iy = re^{i\theta} \quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\left| -\frac{e^{(c+iy)t - xr^{1/3}e^{i\theta/3}}}{re^{i\theta}} \right| \leq \left| \frac{e^{ct - x(c^2+y^2)^{1/6}\cos\frac{\pi}{6}}}{c^2+y^2} \right| \longrightarrow 0 \quad \text{as } |y| \to \infty$$

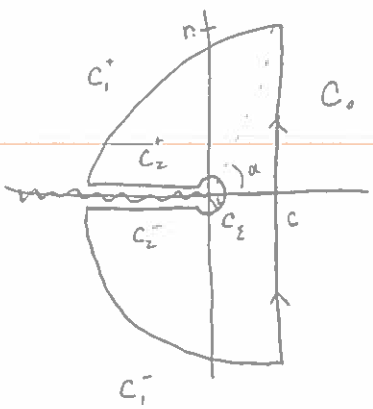
Part 2: $$= \int_{c-i\infty}^{c+i\infty} f(x,s,t)\, t\, ds - \int_{c-i\infty}^{c+i\infty} f(x,s,t) \frac{x}{s^{2/3}}\, ds = ① - ②$$

$$|②| = \left| \int_{c-i\infty}^{c+i\infty} f(x,s,t) \frac{x}{s^{2/3}}\, ds \right| \leq \frac{x}{c^{2/3}} \left| \int_{c-i\infty}^{c+i\infty} f(x,s,t)\, ds \right| \lesssim \cancel{\frac{x}{c^{2/3}}} \left| t \int_{c-i\infty}^{c+i\infty} f(x,s,t)\, ds \right| = |①|$$

↑ as $t \to \infty$

$\Rightarrow |②| \ll |①|$ as $t \to \infty$ so

$$I(x,t) \sim \int_{c-i\infty}^{c+i\infty} \frac{te^{st - xs^{1/3}}}{s}\, ds$$



$$\tilde{C} = C_0 \cup C_1^+ \cup C_1^- \cup C_2^+ \cup C_2^- \cup C_\varepsilon$$

$\tilde{C}$: $$\int_{\tilde{C}} t f(x,s,t)\, ds = 0$$

$\underline{C_1^+}$: $s = re^{i\theta}$ $\quad \theta \in (\alpha, \pi) \begin{cases} \theta \in (\alpha, \frac{\pi}{2}) & r\cos\theta < c \\ \theta \in (\frac{\pi}{2}, \pi) & \cos\theta < 0 \end{cases}$

$ds = ire^{i\theta} d\theta$

$$\left| \int_{C_1^+} t f \, ds \right| \leq \left| \int_\alpha^{\pi/2} t \frac{e^{t\overbrace{r\cos\theta}^{<c} - x r^{1/3} \cos\frac{\theta}{3}}}{\cancel{r}} i\cancel{r} e^{i\theta} d\theta \right|$$

$$+ \left| \int_{\pi/2}^{\pi} \frac{e^{tr\overbrace{\cos\theta}^{<0} - x r^{1/3} \overbrace{\cos\frac{\theta}{3}}^{>0}}}{\cancel{r}} i\cancel{r} e^{i\theta} d\theta \right|$$

$$\leq \int_\alpha^{\pi/2} t \, e^{tc - x r^{1/3} \cos\frac{\theta}{3}} \, d\theta$$

$$+ \int_{\pi/2}^{\alpha} t \, e^{\underbrace{tr\cos\theta - x r^{1/3} \overbrace{\cos\frac{\theta}{3}}^{>0}}_{<0}} \, d\theta$$

$\longrightarrow 0$ as $r \to \infty$

$\underline{C_1^-}$: Similarly as above but $\theta \in (-\alpha, -\pi) \begin{cases} \theta \in (-\alpha, -\frac{\pi}{2}) & r\cos\theta < c \\ \theta \in (-\frac{\pi}{2}, -\pi) & \cos\theta < 0 \end{cases}$

$\underline{C_\varepsilon}$: $s = \varepsilon e^{i\theta}$ $\quad \theta \in (-\pi, \pi)$

$ds = \varepsilon i e^{i\theta} d\theta$

$$t \int_\pi^{-\pi} \cancel{\varepsilon} i \cancel{e^{i\theta}} \frac{e^{t\varepsilon e^{i\theta} - x\varepsilon^{1/3} e^{i\theta/3}}}{\cancel{\varepsilon e^{i\theta}}} d\theta \xrightarrow[\text{as } \varepsilon \to 0]{} -t \int_{-\pi}^{\pi} i \, d\theta = -2\pi i t$$

$\underline{C_2^+:}$ $\quad s = re^{i\pi} \qquad s = -r \qquad r: \infty \longrightarrow 0$

$ds = e^{i\pi}dr \qquad ds = -dr \qquad\qquad \beta = \frac{x\sqrt{3}}{2} r^{1/3}$

$$-\int_\infty^0 t \, \frac{e^{-rt - xr^{1/3}\cos\frac{\pi}{3} - ixr^{1/3}\sin\frac{\pi}{3}}}{-r} \, dr = -\int_0^\infty t \, \frac{e^{-rt - \frac{xr^{1/3}}{2}}}{r} (\cos\beta - i\sin\beta)$$

$\underline{C_2^-:}$ $\quad s = re^{-i\pi} \qquad s = -r \qquad r: 0 \longrightarrow \infty$

$ds = e^{-i\pi}dr \qquad ds = -dr$

$$\int_0^\infty t \, \frac{e^{-rt - \frac{xr^{1/3}}{2}}}{r} (\cos\beta + i\sin\beta) \, dr$$

$$C_2^+ + C_2^- = 2ti \int_0^\infty \frac{e^{-rt - \frac{xr^{1/3}}{2}} \sin\left(\frac{x\sqrt{3}}{2} r^{1/3}\right)}{r} \, dr$$

Use Watson's Lemma

$$e^{-\frac{xr}{2}} \sin\left(\frac{x\sqrt{3}}{2} r\right) \sim \sum_{n=0}^\infty \frac{n\left(\frac{x\sqrt{3}}{2}\right)}{n!} r^n$$

$$\underbrace{\frac{e^{-\frac{xr^{1/3}}{2}} \sin \frac{x\sqrt{3}}{2} r^{1/3}}{r}}_{*} \sim r^{-1} \sum_{n=0}^\infty \underbrace{\frac{\left(\frac{x\sqrt{3}}{2}\right)}{(n-1)!}}_{a_n} r^{n/3} \qquad \alpha = -1 \quad \beta = \frac{1}{3}$$

$$2ti \int_0^\infty * \, e^{-rt} \, dr \sim 2ti \, \frac{\frac{x\sqrt{3}}{2} \Gamma\left(-1 + \frac{1}{3}(0) + 1\right)}{t^{\left(-1 + \frac{1}{3}(0) + 1\right)}} = \sqrt{3} t x i \, \Gamma(0) = \sqrt{3} t x i$$

Watson's Lemma

c: If $r \to \infty$ and $\varepsilon \to 0$

$$t \int_{c-i\infty}^{c+i\infty} \frac{e^{st - xs^{1/3}}}{s}\, ds \sim 2\pi i t - \sqrt{3}\, t x i$$

$$I(x,t) \sim \frac{1}{2\pi i}\left(\qquad \right)$$

$$\boxed{I(x,t) \sim t - \frac{\sqrt{3}}{2\pi} x t \qquad \text{as } t \to \infty}$$