METHODS OF APPLIED MATHEMATICS COMPREHENSIVE EXAMINATION JANUARY 2016

Work on as many of the following problems as possible. Turn in *all* your work.

(1) Consider two bodies of mass m_1 and m_2, respectively, joined by a spring of constant k, in collinear motion along the z-axis of a cartesian frame in $\mathbb{R}^3$. If the position of the first body is at $z = r_1(t)$ and the second is at $z = r_2(t)$, write evolution equations given by Newton's second law for these functions of time t, when both bodies are immersed in a fluid which exerts a resistance force proportional to the bodies velocities $F_j \equiv -\mu \dot{r}_j$, $j = 1, 2$, with constant $\mu > 0$. The bodies are also subject to the gravitational force field $F_g = -GM_e m_j/z^2$ (with negligible mutual gravitational attraction between the two bodies), where $z = 0$ corresponds to the Earth's center, M_e and m_j are the Earth's and body's mass, respectively, and G is the universal gravitational constant.

(a) Non-dimensionalize the equations of motion; assume $\epsilon \equiv m_2/m_1 \ll 1$, with $M_e \gg m_1$, and that the initial separation h between the two bodies is $0 < h \equiv r_1(0) - r_2(0)$, with $h \ll r_1(0)$. Discuss all the possible dominant balances for the dynamics.

(b) Neglecting gravity, find the solution of these equations corresponding to zero initial velocities.

(c) Write an asymptotic expansion of the equations of motion, and find the leading order terms of their solutions in the absence of gravity.

(d) Find the leading order terms of the motion equations in the Earth gravitational field, and sketch the leading order solutions noting their time scale of validity.

(2) Consider the following function of the complex variable z in the complex plane:

$$f(z) = \frac{1}{\sqrt{z^2+1}} \log\left(\frac{1-\sqrt{z^2+1}}{1+\sqrt{z^2+1}}\right)$$

(a) Classify all singularities and propose branch cuts, if necessary, to make the function single valued on appropriate domains.

(b) Discuss the convergence of the real integral

$$\int_{-1}^{1} \frac{1}{\sqrt{|a^2-t^2|}} \arctan\left(\frac{1}{\sqrt{1-t^2}}\right) dt$$

as a function of the real parameter a.

(c) Propose a strategy for computing the real integral when $a = 1$ based on the study of the function $f(z)$. Discuss your proposal even if you cannot carry out all the steps to evaluate the value of the integral, if it is finite.

(3) Consider the rapidly varying diffusivity:

$$K(x, y; \epsilon) = A + F(x/\epsilon^2) + G(y/\epsilon)$$

where A is chosen to guarantee K is positive, and ϵ is a small constant. By applying iterated homogenization, average the following diffusion equation subject to the rapidly varying "potential" $V(x/\varepsilon) > 0$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K(x,y;\epsilon)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K(x,y;\epsilon)\frac{\partial u}{\partial y}\right) + V\left(\frac{x}{\varepsilon}\right)u$$

$$u(x,y,0) = u_0(x,y)\,,$$

by computing a leading order effective equation governing the evolution as $\epsilon \to 0$ over the (x,y)-plane, assuming the functions $F(x)$, $V(x)$ and $G(y)$ are mean zero, periodic, and share the same period. Solve the averaged equation in free space.

(4) Consider the 6th-order ordinary differential equation on the real line $x \in \mathbb{R}$,

$$\frac{d^6 y}{dx^6} = x\,y + 4y\,.$$

(a) By using a contour integral representation in the appropriate complex Laplace-image plane, discuss whether solutions decaying as $|x| \to \infty$ exist.

(b) Find the leading order asymptotic expansion of one of these solutions as $x \to \infty$ by appropriate deformations of the contour integral representation.

(5) Find two term asymptotic expansions as $\epsilon \to 0$ for all roots of the equation:

$$\epsilon^2 z^4 + az^3 + 3z^2 - 3z + 1 = \epsilon$$

Explicitly handle and discuss special values of a.

(6) Consider the following initial value problem for the time-t evolution equation in one spatial dimension $x \in \mathbb{R}$

$$T_t + \gamma\, x\, T_x = \kappa T_{xx}\,, \qquad T(x,0) = T_0(\alpha x)\,.$$

(a) What are the units of the parameters γ, κ and α?

(b) Non-dimensionalize the equation.

(c) Use the Fourier-transform method with definition

$$\hat{T}(k,t) \equiv \int_{-\infty}^{\infty} T(x,t)e^{-ikx}\,dx$$

to solve the resulting equation for $\hat{T}$ by characteristics.

(d) What PDE does $u \equiv T_x$ solve?

(e) Compare the long-time asymptotics of T vs. u assuming that T_0 is a Heaviside step function.

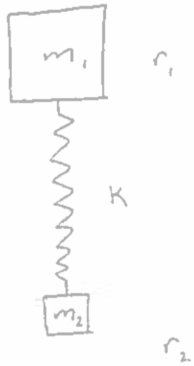
(7) (a) Explain the difference between pointwise convergence and asymptotic convergence. Illustrate with the particular example of power series.

(b) What does it take for a Laurent series around a singular point to be an asymptotic series? Discuss.

Problem 1

Jan 2016

a)



$$\begin{cases} m_1 \ddot{r}_1 = -K(r_1 - r_2 - d) - \mu \dot{r}_1 - \dfrac{G M_e m_1}{r_1^2} & (1) \\ m_2 \ddot{r}_2 = K(r_1 - r_2 - d) - \mu \dot{r}_2 - \dfrac{G M_e m_2}{r_2^2} & (2) \end{cases}$$

where d is the length of the spring at rest.

Assume $m_2 \ll m_1$ so $\varepsilon = m_2/m_1$.

Let $L = d$ and $T = \sqrt{\frac{m_2}{K}}$ and define

$$r_i = z_i L \quad \text{and} \quad t = \tau T$$

non-dim

$$\Rightarrow \frac{dr_i}{dt} = L \frac{dz_i}{dt} = \frac{L}{T} \frac{dz_i}{d\tau} = \frac{L}{T} \dot{z}_i$$

$$\frac{d^2 r_i}{dt^2} = \frac{L}{T^2} \ddot{z}_i$$

Then (1) and (2) become

$$\begin{cases} m_1 \frac{L}{T^2} \ddot{z}_1 = -KL(z_1 - z_2 - 1) - \mu L \dot{z}_1 - \dfrac{G M_e m_1}{L^2 z_1^2} \\ \text{" similar "} \end{cases}$$

$$\begin{cases} \ddot{z}_1 = \frac{m_2}{m_1}(z_1 - z_2 - 1) - \frac{\mu T^2}{m_1} \dot{z}_1 - \dfrac{G M_e T^2}{L^3 z_1^2} \\ \text{" similar "} \end{cases}$$

$$\boxed{\begin{cases} \ddot{z}_1 = \varepsilon(z_1 - z_2 - 1) - \frac{\mu}{K} \varepsilon \dot{z}_1 - \dfrac{G M_e T^2}{L^3 z_1^2} & (3) \\ \ddot{z}_2 = z_1 - z_2 - 1 - \frac{\mu}{K} \dot{z}_2 - \dfrac{G M_e T^2}{L^3 z_2^2} & (4) \end{cases}}$$

b) Neglecting gravity:

$$
\begin{cases}
\ddot{z}_1 = \varepsilon(z_1 - z_2 - 1) - \frac{\mu}{k}\varepsilon \dot{z}_1 & (5) \\
\ddot{z}_2 = z_1 - z_2 - 1 - \frac{\mu}{k}\dot{z}_2 & (6)
\end{cases}
\quad \Rightarrow \quad z_1 = \ddot{z}_2 + \frac{\mu}{k}\dot{z}_2 + z_2 + 1
$$

Plug into (5):

$$\ddddot{z}_2 + \frac{\mu}{k}\dddot{z}_2 + \ddot{z}_2 = \varepsilon\left(\ddot{z}_2 + \frac{\mu}{k}\dot{z}_2\right) - \frac{\mu}{k}\varepsilon\left(\dddot{z}_2 + \frac{\mu}{k}\ddot{z}_2 + \dot{z}_2\right)$$

$$\ddddot{z}_2 + (1+\varepsilon)\frac{\mu}{k}\dddot{z}_2 + \left(1 - \varepsilon + \frac{\mu}{k}\right)\ddot{z}_2 + \left(1 - \frac{\mu}{k}\right)\dot{z}_2 = 0$$

Problem 2

Jan 2016

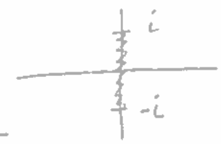
$$f(z) = \frac{1}{\sqrt{z^2+1}} \log\left(\frac{1-\sqrt{z^2+1}}{1+\sqrt{z^2+1}}\right)$$

a) Branch Points

$\sqrt{\ }$: $z^2+1=0$

$z = \pm i$

Use branch cut :



and positive branch of $\sqrt{\ }$

log: BP's occur at 0 and ∞ so...

0: $1-\sqrt{z^2+1} = 0$ $\qquad$ ∞: $1+\sqrt{z^2+1} = 0$

$z = 0$ $\qquad\qquad$ $z = 0$

Now we must check if $z=0$ is actually a b.p.

To do this let's use polar notation:

$$\log\left(\frac{1-\sqrt{\rho_1\rho_2}\, e^{\frac{\theta_1+\theta_2}{2}i}}{1+\sqrt{\rho_1\rho_2}\, e^{\frac{\theta_1+\theta_2}{2}i}}\right) \stackrel{?}{=} \log\left(\frac{1-\sqrt{\rho_1\rho_2}\, e^{\frac{(\theta_1+2\pi)+(\theta_2+2\pi)}{2}i}}{1+\sqrt{\rho_1\rho_2}\, e^{\frac{(\theta_1+2\pi)+(\theta_2+2\pi)}{2}i}}\right)$$

$$\ln\Big(\underbrace{1-\sqrt{\rho_1\rho_2}\, e^{\frac{\theta_1+\theta_2}{2}i}}_{1}\Big) - \ln\Big(\underbrace{1+\sqrt{\rho_1\rho_2}\, e^{\frac{\theta_1+\theta_2}{2}i}}_{2}\Big) \stackrel{?}{=} \underbrace{\text{expand}}_{3} \text{ that } \underbrace{\uparrow}_{4}$$

$$\cancel{\ln(\text{stuff})} + \cancel{i}(\arg(1)-\arg(2)) \stackrel{?}{=} \cancel{\ln(\text{same stuff})} + \cancel{i}(\arg(3)-\arg(4))$$

$$\phi_1 - \phi_2 = \phi_1 + 2\pi - \phi_2 - 2\pi \quad \checkmark$$

Since they are equal, $z=0$ is not a bp.

Thus $z=\pm i$ are the only bp's and also essential singularities.

Problem 3

Jan 2011

$$K = A + F\left(\frac{x}{\varepsilon^2}\right) + G\left(\frac{y}{\varepsilon}\right)$$

$$u_t = \partial_x(K\partial_x u) + \partial_y(K\partial_y u) + v\left(\frac{x}{\varepsilon}\right)u$$

$$u(x,y,0) = u_0(x,y)$$

Let $w = \frac{x}{\varepsilon^2}$ and $z = \frac{y}{\varepsilon}$

$\Rightarrow \quad \partial_x \mapsto \partial_x + \frac{1}{\varepsilon}\partial_z \quad$ and $\quad \partial_y \mapsto \partial_y + \frac{1}{\varepsilon^2}\partial_w$

Ansantz: $u = \bar{u} + \varepsilon u_1 + \ldots$

$$u_t = \left(\partial_x + \tfrac{1}{\varepsilon^2}\partial_w\right)\left[K\left(\partial_x + \tfrac{1}{\varepsilon^2}\partial_w\right)u\right] + \left(\partial_y + \tfrac{1}{\varepsilon}\partial_z\right)\left[K\left(\partial_y + \tfrac{1}{\varepsilon}\partial_z\right)u\right] + v(z)u$$

$O\left(\frac{1}{\varepsilon^4}\right)$: $\partial_w(K\partial_w\bar{u}) = 0 \quad \Rightarrow \quad \bar{u}$ indep w

$O\left(\frac{1}{\varepsilon^3}\right)$: $\partial_w(K\partial_w u_1) = 0 \quad \Rightarrow \quad u_1$ indep w

$O\left(\frac{1}{\varepsilon^2}\right)$: $\partial_w(K\partial_w u_2) + \cancelto{0}{\partial_x(K\partial_w\bar{u})} + \partial_w(K\partial_x\bar{u}) + \partial_z(K\partial_z\bar{u}) = 0$

$$\cancelto{0}{\langle\partial_w(K\partial_w u_2 + K\partial_x\bar{u})\rangle_w} + \langle\partial_z K\partial_z\bar{u}\rangle_w = 0$$

FA

$$\partial_z\langle K\rangle_z\partial_z\bar{u} = 0$$

$\Rightarrow \bar{u}$ indep z

$$K\partial_w u_2 + K\partial_x\bar{u} = A(x,y,z,t)$$

$$\langle\cancelto{0}{\partial_w u_2}\rangle_w + \langle\partial_x\bar{u}\rangle_w = \left\langle\frac{A}{K}\right\rangle_w$$

$$\partial_x\bar{u} = A\left(\langle K\rangle_w^h\right)^{-1} \quad \Rightarrow \quad A = \langle K\rangle_w^h\,\partial_x\bar{u}$$

$$K(\partial_w u_2 + K\partial_x\bar{u}) = \langle K\rangle_w^h\,\partial_x\bar{u}$$

$$O\left(\tfrac{1}{\varepsilon}\right): \partial_w(K\partial_w u_3) + \cancelto{0}{\partial_x(K\partial_w u_1)} + \partial_w(K\partial_x u_1) + \partial_z(K\partial_z u_1)$$

$$+ \cancelto{0}{\partial_y(K\partial_z \bar{u})} + \partial_z(K\partial_y \bar{u}) = 0$$

$$\underbrace{\cancelto{0}{\langle \partial_w(K\partial_w u_3 + K\partial_x u_1)\rangle_w}}_{FA} + \langle \partial_z(K\partial_z u_1 + K\partial_y \bar{u})\rangle_w = 0$$

$$\langle K\rangle_w \partial_z u_1 + \langle K\rangle_w \partial_y \bar{u} = B(x,y,w,t)$$

$$\langle \partial_z u_1\rangle_z + \langle \partial_y \bar{u}\rangle_z = \left\langle \frac{B}{\langle K\rangle_w}\right\rangle_z$$

$$\partial_y \bar{u} = B\left(\langle\langle K\rangle_w\rangle_z^h\right)^{-1}$$

$$\Rightarrow \langle K\rangle_w \partial_z u_1 + \langle K\rangle_w \partial_y \bar{u} = \langle\langle K\rangle_w\rangle_z^h \partial \bar{u}$$

$$O(1): \partial_w(K\partial_w u_4) + \partial_x(K\partial_w u_2) + \partial_w(K\partial_x u_2) + \partial_z(K\partial_z u_2) + \partial_y(K\partial_z u_1)$$

$$+ \partial_z(K\partial_y u_1) + \partial_x(K\partial_x \bar{u}) + \partial_y(K\partial_y \bar{u}) + V(z)\bar{u} = \bar{u}_t$$

$$\partial_w(K\partial_w u_4) + \partial_x\big(\overbrace{K\partial_w u_2 + K\partial_x \bar{u}}^{\langle K\rangle_w^h \partial_x \bar{u}}\big) + \partial_w(K\partial_x u_2) + \partial_z(K\partial_z u_2) + \partial_z(K\partial_y u_1)$$

$$+ \partial_y(K\partial_z u_1 + K\partial_y \bar{u}) + V(z)\bar{u} = \bar{u}_t$$

$$\cancelto{0}{\langle \partial_w(K\partial_w u_4)\rangle_w} + \langle \langle K\rangle_w^h \bar{u}_{xx}\rangle_w + \cancelto{0}{\langle \partial_w(K\partial_x u_2)\rangle_w} + \langle \partial_z(K\partial_z u_2 + K\partial_y u_1)\rangle_w$$

$$\langle \partial_y(K\partial_z u_1 + K\partial_y \bar{u})\rangle_w + \langle V(z)\bar{u}\rangle_w = \langle \bar{u}_t\rangle_w$$

$$\langle \langle K\rangle_w^h \bar{u}_{xx}\rangle_z + \cancelto{0}{\langle \partial_z(\langle K\rangle_w \partial_z u_2 + \langle K\rangle_w \partial_y u_1)\rangle_z} + \cancelto{0}{\langle V(z)\bar{u}\rangle_z} = \langle \bar{u}_t\rangle_z$$

$$+ \langle \langle\langle K\rangle_w\rangle_z^h \bar{u}_{yy}\rangle_z$$

periodic

indep of z

$$\bar{u}_t = \underbrace{\langle\langle K\rangle_w^h\rangle_z}_{\alpha} \bar{u}_{xx} + \underbrace{\langle\langle K\rangle_w\rangle_z^h}_{\beta} \bar{u}_{yy}$$

$$\boxed{\bar{u}(x,y,t) = \mathcal{F}^{-1}\left\{c e^{(\alpha \chi^2 + \beta \eta^2)t}\right\}}$$

where $\mathcal{F}(\bar{u}_{xx}) = \chi^2$

$\mathcal{F}(\bar{u}_{yy}) = \eta^2$

Problem 4

Jan 2016

$$y^{(6)} = xy + 4y \qquad (1)$$

a) Let $y = f(x) = \int_C F(s) e^{sx}\, ds$ for some path C. (2)

$$\Rightarrow \quad y^{(6)} = \int_C F(s)\, s^6 e^{sx}\, ds$$

$$xy = \int_C F(s)\, x e^{sx}\, ds = \cancel{e^{sx}F(s)\Big|_C}^{\,0} - \int_C F'(s) e^{sx}\, ds$$

Conditions on C

- convergent integrals
- $e^{sx}F(s)\Big|_C = 0$

Plug into (1):

$$\int_C \left(s^6 F(s) + F'(s) - 4F(s)\right) e^{sx}\, ds = 0$$

$$F(s)\left(s^6 - 4\right) = -F'(s)$$

$$F'(s) = -F(s)\left(s^6 - 4\right)$$

$$F(s) = a e^{-\frac{s^7}{7} + 4s} \qquad \text{where } s = re^{i\theta}$$

Plug into (2):

$$y = a\int_C e^{-\frac{s^7}{7} + 4s + sx}\, ds$$

For this to decay, we need $y \to 0$ as $x \to \infty$.

This occurs when $Re(s) > 1 \because s^7$ will dominate s.

b) We know $s^7 = r^7 e^{7i\theta}$. So $Re(s^7) > 1$ when $\cos 7\theta > 0$

$$-\frac{\pi}{2} + 2\pi k < 7\theta < \frac{\pi}{2} + 2\pi k$$

$$-\frac{\pi}{14} + \frac{2\pi k}{7} < \theta < \frac{\pi}{14} + \frac{2\pi k}{7}$$

$$-\frac{\pi}{14} < \theta < \frac{\pi}{14}$$

$$\frac{3\pi}{14} < \theta < \frac{5\pi}{14}$$

$$\frac{\pi}{2} < \theta < \frac{9\pi}{14}$$

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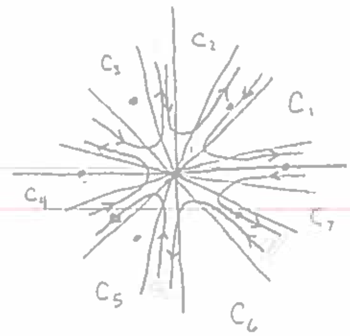

Choose γ st y decays as $x \to \infty$ and sps $s \sim x^\alpha \xi$.

$$f(x) = \int_{C_i} e^{-\frac{s^7}{7} + (4+x)s}\, ds = \int_{C_i} x^\alpha e^{-\frac{x^{7\alpha}\xi^7}{7} + (4+x)x^\alpha \xi}\, d\xi$$

as $x \to \infty$ 4 is negligable

$$\approx \int_{C_i} x^{1/6} e^{x^{7/6}(-\xi^7/7 + \xi)}\, d\xi$$

$7\alpha = \alpha + 1$

$\alpha = 1/6$

$h(\xi) = -\frac{1}{7}\xi^7 + \xi$

$h'(\xi) = -\xi^6 + 1 = 0$ $\quad \xi^* = e^{\frac{2\pi i}{6}k}$ $\quad k = 0, 1, \ldots, 5$ $\quad$ use $\xi^* = -1$ $\because$ easy

$h''(\xi) = -6\xi^5$

$h''(\xi^*) = 6 \neq 0$

$= 6e^{i0}$ $\quad \Rightarrow \quad \alpha = 0$

Want $Re(h(s)) < 0$ and $Im(h(s)) = 0$

$\cos(\alpha + 2\theta_m) < 0$

$\sin(\alpha + 2\theta_m) = 0$

$\alpha + 2\theta_m = m\pi \qquad m = 0, 1$

$\theta_m = \frac{m\pi}{2}$

$\theta_0 = 0 \qquad \theta_1 = \frac{\pi}{2}$

$\theta_0: \cos(0) = 1 \not< 0$

$\theta_1: \cos(\pi) = -1 < 0$

$\theta = \pi/2$

$$y = \int_{C_4} x^{1/6} e^{x^{7/6}(-\xi^7/7 + \xi)} dk \sim (\xi^*)^{1/6} e^{xh(\xi^*)} \sqrt{\frac{2\pi}{x|h''(\xi^*)|}} e^{i\theta}$$

$h(\xi^*) = 1/7 - 1 = -6/7$

$h''(\xi^*) = 6$

$$y \sim e^{\frac{\pi i}{6}} e^{-6/7 x} \sqrt{\frac{\pi}{3x}} e^{i\pi/2}$$

$$\boxed{y \sim i\, e^{-6/7 x + i\frac{\pi}{6}} \sqrt{\frac{\pi}{3x}}}$$

Problem 4

Jan 2016

$$y^{(6)} = xy + 4y \qquad (1)$$

a) For some path C:

$$y = f(x) = \int_c F(s)\, e^{sx}\, ds \qquad (2)$$

$$\Rightarrow f^{(6)}(x) = \int_c F(s)\, s^6 e^{sx}\, ds$$

$$x f(x) = \int_c F(s)\, x e^{sx}\, ds = e^{sx}F(s)\Big|_c - \int_c F'(s) e^{sx}\, ds = -\int_c F'(s) e^{sx}\, ds$$

$u = F(s)$ $\quad$ $v = e^{sx}$

$du = F'(s)\,ds$ $\quad$ $dv = x e^{sx}\, ds$

Conditions on C

- convergent integrals
- $e^{sx}F(s)\big|_c = 0$

Plug into (1):

$$\Rightarrow \int_c \left(F(s)s^6 + F'(s) - 4F(s)\right) e^{sx}\, ds = 0$$

$$F(s)s^6 - 4F(s) = -F'(s)$$

$$-(s^6 - 4)\, F(s) = F'(s)$$

$$F(s) = a\, e^{-s^7/7 - 4s} \qquad \text{where} \qquad s = re^{i\theta}$$

Plug into 2:

$$f(x) = a \int_c e^{-\frac{s^7}{7} + 4s + sx}\, dx$$

For this to decay we need $f(x) \to 0$ as $x \to \infty$

which occurs when $Re(s^7) > 0$ $\because$ s^7 will dominate s.

b) Let $s^7 = r^7 e^{7i\theta}$. Then $Re(s^7) > 0$ if $\cos 7\theta > 0$.

$-\frac{\pi}{2} + 2\pi k < 7\theta < +\frac{\pi}{2} + 2\pi k$

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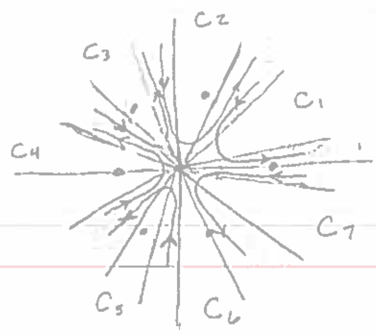

Choose γ st y decays as $x \to \infty$

$$f(x) = \int_{C_i} e^{-\frac{s^7}{7} + (4+x)s} dx = \int_{C_i} x^\alpha e^{-\frac{x^{7\alpha}\xi^7}{7} + (4+x)x^\alpha \xi} d\xi$$

as $x \to \infty$ 4 is negligable

$s = x^\alpha \xi \Rightarrow ds = x^\alpha d\xi$

$$\approx \int_{C_i} x^\alpha e^{-\frac{1}{7}x^{7\alpha}\xi^7 + x^{\alpha+1}\xi} d\xi \qquad 7\alpha = \alpha + 1, \quad \alpha = \frac{1}{6}$$

$$= \int_{C_i} x^{1/6} e^{x^{7/6}(-\frac{1}{7}\xi^7 + \xi)} d\xi$$

$h(\xi) = -\frac{1}{7}\xi^7 + \xi$

$h'(\xi) = 1 - \xi^6 = 0 \Rightarrow \xi^* = e^{\frac{2\pi i}{6}k}$ $\quad k = 0, \dots, 5$ $\quad$ use $\xi^* = -1$ $\because$ easy

$h''(\xi) = -6\xi^5 \neq 0 \quad \forall k \Rightarrow n = 2$

$h''(-1) = -6(-1)^5 = 6 \Rightarrow \alpha = 0$

$\Rightarrow \theta_0 = -\frac{0}{2} + \frac{\pi}{2} = \frac{\pi}{2}$

$\theta_1 = 0 + (2\cdot 1 + 1)\frac{\pi}{2} = \frac{3\pi}{2}$

$$y = f(x) = \int_{C_4} x^{1/6} e^{x^{7/6} h(\xi)} d\xi = x^{1/6} \int_{C_4} e^{x^{7/6} h(\xi)} d\xi$$

$$\sim x^{1/6} e^{x\, h(-1)} \left(e^{i\frac{\pi}{2}} - e^{i\frac{3\pi}{2}}\right) \frac{\Gamma(1/2)}{\left(x^{7/6} |h''(-1)|\right)^{1/2}} \frac{(2!)^{1/2}}{2}$$

as $x \to \infty$

$$y \sim x^{1/6} e^{0} \left(e^{i\frac{\pi}{2}} - e^{i\frac{3\pi}{2}}\right) \frac{\Gamma(1/2)}{(6x^{7/6})^{1/2}} \frac{1}{2^{1/2}}$$

$$\boxed{y \sim x^{1/6} \left(e^{i\frac{\pi}{2}} - e^{i\frac{3\pi}{2}}\right) \frac{\Gamma(1/2)}{2(3x^{7/6})^{1/2}}}$$

Problem 5

$$\varepsilon^2 z^4 + a z^3 + 3z^2 - 3z + 1 = \varepsilon \qquad (1)$$

Consider the unpreturbed roots:

$$a z^3 + 3z^2 - 3z + 1 = 0 \qquad (2)$$

If $a = -1$ then

$$z^3 - 3z^2 + 3z - 1 = 0 \qquad (3)$$

$$(z-1)^3 = 0$$

$$z_0 = 1 \quad \text{degen root}$$

$$z_i = \sum_{n=0}^{\infty} b_n^i \, \varepsilon^{n/3} \qquad \text{where } b_0 = 1 \text{ and } i = 1, 2, 3$$

Plug z_i into (1) w/ $a = -1$:

$$\varepsilon^2 (1 + b\varepsilon^{1/3} + \dots)^4 - (1 + b\varepsilon^{1/3} + \dots - 1)^3 = \varepsilon$$

$O(\varepsilon)$: $b^3 = 1$.

$b = \omega$ where $\omega^3 = 1$

$$\boxed{z_1 \sim 1 + \omega \varepsilon^{1/3} \qquad z_2 \sim 1 + \omega^2 \varepsilon^{1/3} \qquad z_3 \sim 1 + \varepsilon^{1/3}}$$

Let $x = \varepsilon^{\alpha} z \Rightarrow z = \varepsilon^{-\alpha} x$ and plug into (1):

$$\varepsilon^2 (\varepsilon^{-4\alpha} x^4) - \varepsilon^{-3\alpha} x^3 + 3\varepsilon^{-2\alpha} x^2 - 3\varepsilon^{-\alpha} x + 1 = \varepsilon$$

$$\varepsilon^{2-4\alpha} x^4 - \varepsilon^{-3\alpha} x^3 + 3\varepsilon^{-2\alpha} x^2 - 3\varepsilon^{-\alpha} x + 1 = \varepsilon \qquad (4)$$

$$2 - 4\alpha = -3\alpha$$

$$2 = \alpha$$

$$\varepsilon^{-6}x^4 - \varepsilon^{-6}x^3 + 3\varepsilon^{-4}x^2 - 3\varepsilon^{-2}x + 1 = \varepsilon$$

$$x^4 - x^3 + 3\varepsilon^2 x^2 - 3\varepsilon^4 x + \varepsilon^6 = \varepsilon^7 \tag{5}$$

If $\varepsilon = 0$ then:

$$x^4 - x^3 = 0$$

$$x^3(x-1) = 0$$

$$x = 0, 0, 0, 1$$

correspond to z_1, z_2, z_3 roots

$$x_4 = 1 + b_1\varepsilon + b_2\varepsilon^2 + \dots$$

sub into (5):

$$(1+b_1\varepsilon+\dots)^4 - (1+b_1\varepsilon+\dots)^3 + 3\varepsilon^2(1+b_1\varepsilon+\dots)^2 - 3\varepsilon^4(1+b_1\varepsilon+\dots) + \varepsilon^6 = \varepsilon^7$$

$O(1)$: $1 - 1 = 0$ ✓

$O(\varepsilon)$: $4b_1 - 3b_1 = 0$

$$b_1 = 0$$

$O(\varepsilon^2)$: $\cancel{6b_1^2} + 4b_2 - \cancel{6b_1^2} - 3b_2 + 3 = 0$

$$b_2 = -3$$

$$x_4 \sim 1 - 3\varepsilon^2$$

$$z_4 \sim \varepsilon^{-2}(1 - 3\varepsilon^2)$$

$$\boxed{z_4 \sim \frac{1}{\varepsilon^2} - 3}$$

Problem 6

$$\begin{cases} T_t + \gamma x\, T_x = k\, T_{xx} \\ T(x,0) = T_0(\alpha x) \end{cases} \qquad (1)$$

a) $\boxed{[\alpha] = \frac{1}{L}}$

Let $x = Lz$ and $t = T\tau$

$\Rightarrow \quad T_x = \frac{1}{L} T_z$

$T_t = \frac{1}{T} T_\tau$

To non-dim (1) we get:

$$\frac{1}{T} T_\tau + \gamma L z \left(\frac{1}{L}\right) T_z = k \frac{1}{L^2} T_{zz}$$

$$T_\tau + \gamma T z T_z = k \frac{T}{L^2} T_{zz} \qquad (2)$$

$\Rightarrow \boxed{\begin{array}{l} [\gamma] = \frac{1}{T} \\ [k] = \frac{L^2}{T} \end{array}}$ to make (2) non-dim

b) Let $\bar{\gamma} = \gamma T$ and $\bar{k} = k \frac{T}{L^2}$ (note these are dimensionless parameters).

$\Rightarrow \quad T_\tau + \bar{\gamma} z T_z = \bar{k} T_{zz}$ is non-dim

For ease of notation, we can use the origonal notation and assume that it is non-dim.

$$\begin{cases} T_t + \gamma x\, T_x = k\, T_{xx} \\ T(x,0) = T_0(x) \end{cases}$$

c) $$\widehat{xT_x} = \int_{-\infty}^{\infty} T_x \, x e^{-ikx} \, dx = i \frac{d}{dk} \int_{-\infty}^{\infty} T_x e^{-ikx} \, dx$$

$$= i \frac{d}{dk} \left(\cancel{e^{-ikx} T \Big|_{-\infty}^{\infty}} \xrightarrow{} 0 + ik \int_{-\infty}^{\infty} T e^{-ikx} \, dx \right)$$

$$= i^2 \frac{d}{dk} (k \hat{T}) = - \hat{T} - k \hat{T}_k$$

$$\widehat{T_t} = \hat{T}_t$$

$$\widehat{T_{xx}} = (-ik)^2 \hat{T} = -k^2 \hat{T}$$

Thus $\mathcal{F}\{(1)\}$ gives:

$$\hat{T}_t + \gamma(-\hat{T} - k\hat{T}_k) = -Rk^2 \hat{T}$$

$$\hat{T}_t - \gamma k \hat{T}_k = \gamma \hat{T} - Rk^2 \hat{T} \qquad (3)$$

<u>Method of Characteristics</u>

$$\begin{cases} z(s) = \hat{T}(K(s), \Upsilon(s)) \\ K(0) = k \\ \Upsilon(0) = t \end{cases}$$

$$z_s = \hat{T}_K \frac{dK}{ds} + \hat{T}_\Upsilon \frac{d\Upsilon}{ds} \qquad \text{by defn}$$

$$z_s = (\gamma - RK^2(s))z \qquad \text{by eqn (3)} \qquad (4)$$

Using the LHS of (3), we know:

$$\begin{cases} \frac{dK}{ds} = -\gamma K(s) \\ \frac{d\Upsilon}{dt} = 1 \end{cases} \Rightarrow \begin{cases} K(s) = k e^{-\gamma s} \\ \Upsilon(s) = s + t \end{cases}$$

Thus eqn (4) becomes:

$$z_s = (\gamma - Rk e^{-\gamma s}) z$$

$$\Rightarrow \frac{dz}{z} = (\gamma - Rk e^{-\gamma s})\, ds$$

$$z(s) = z(0)\, e^{\int_0^s (\gamma - Rk e^{-\gamma v})\, dv}$$

Note $z(0) = \hat{T}(K(0), T(0)) = \hat{T}(k, t)$

$$z(0) = z(s)\, e^{-(s + \frac{Rk}{\gamma}(e^{-\gamma s} - 1))}$$

If $s = -t$ then $z(s) = z(-t) = \hat{T}(k e^{\gamma t}, 0)$

$$= \hat{T}_0(k e^{\gamma t})$$

where $\hat{T}_0(k) = \int_{-\infty}^{\infty} e^{-ikx}\, T_0(x)\, dx$.

So let $s = -t$.

$$\Rightarrow \boxed{T(x,t) = \int_{\mathbb{R}} e^{ikx}\, \hat{T}_0(k e^{\gamma t})\, e^{t - \frac{Rk}{\gamma}(e^{\gamma t} - 1)}\, dk}$$

d) Take deriv of (1) wrt x:

$$T_{xt} + \gamma (T_x + x T_{xx}) = R\, T_{xx}$$

Let $u = T_x$ then:

$$u_t + \gamma(u + x u_x) = R u_x \qquad (5)$$

(5) is v. similar to (3).

e) Fourier trans gives

$$\hat{u}_t + \gamma \hat{u} - \hat{u} - k \hat{u}_k = R k i \hat{u}$$

$$\hat{u}_t - k \hat{u}_k = (1 - \gamma + i R k) \hat{u} \qquad (6)$$

MoC:

$$\begin{cases} z(s) = \hat{u}(K(s), \Upsilon(s)) \\ K(0) = k \\ \Upsilon(0) = t \end{cases} \Rightarrow z_s = \hat{u}_K \frac{dK}{ds} + \hat{u}_T \frac{d\Upsilon}{ds}$$

$$z_s = (1 - \gamma + i R K(s))\, z \qquad \text{by (6)}$$

$$z(s) = z(0)\, e^{\int_0^s (1 - \gamma + i R K(v))\, dv}$$

$$\begin{cases} \frac{dK}{ds} = -K(s) \\ \frac{d\Upsilon}{ds} = 1 \end{cases} \Rightarrow \begin{cases} K(s) = k e^{-s} \\ \Upsilon(s) = s + t \end{cases}$$

$$\Rightarrow z(0) = z(s)\, e^{-\int_0^s (1 - \gamma + i R k e^{-v})\, dv}$$

$$= z(s)\, e^{-(1-\gamma)s + iRk(e^{-s} - 1)} \qquad \text{let } s = -t$$

$$\boxed{u(x,s) = \int_{\mathbb{R}} e^{ikx}\, \hat{u}_0(k e^{t})\, e^{(1-\gamma)t}\, e^{iRk(e^{t} - 1)}\, dk}$$

Jan 2016

Problem 6

$$T_t + \gamma x T = \bar{\kappa} T_{xx}$$

$$T(x,0) = T_0(\alpha x)$$

c) $\hat{u}_t + \gamma \widehat{(zu)} = \bar{\kappa}(ik)^2 \hat{u}$ where z and τ are non-dim

$$\widehat{xu_x} = \int_R z e^{-ikz} u_z(z,\tau)\,dz = (-1/i)\int_R \frac{\partial}{\partial k} e^{-ikz} u_z(z,\tau)\,dz$$

$$= i\frac{\partial}{\partial k}\int_R e^{-ikz} u_z(z,\tau)\,dz$$

$$= i\frac{\partial}{\partial k}\Big(e^{-ikz} u(z,\tau)\,dz$$

$$\hat{u} = \dots = -\frac{\partial}{\partial k}(k\hat{u})$$

$$z(s) = \hat{u}(k(s), \tau(s))$$

$$\frac{dz}{ds} = \frac{\partial \hat{u}}{\partial k}\frac{\partial k}{\partial s} + \frac{\partial \hat{u}}{\partial \tau}\frac{d\tau}{ds} = \left(\bar{\kappa}(-k(s)^2) + \gamma\right) z$$

$$\frac{\partial z}{z} = \left(\bar{\kappa}(-k(s)^2) + \gamma\right) ds$$

$$\frac{\partial k}{\partial s} = -\gamma k \qquad k(0) = k$$

$$\Rightarrow \frac{dk}{k} = -\gamma\, ds \Rightarrow k(s) = e^{-\gamma s} k$$

$$\frac{d\tau}{ds} = 1 \qquad \tau(0) = t \qquad \Rightarrow \tau(s) = s + t$$

$$z(s) = z(0) e^{\int_0^s (-\bar{\gamma}(ke^{-\gamma v})^2 + \gamma)\,dv} \qquad = -\bar{\kappa} k^2 \left(-\tfrac{1}{2\gamma}\right) e^{-2v\gamma} + \gamma v \Big|_0^s$$

$$= \left(-\bar{\kappa} k^2\left(-\tfrac{1}{2\gamma}\right) e^{-2\gamma s} + \gamma s\right) + \bar{\kappa} k^2\left(-\tfrac{1}{2\gamma}\right)$$

$$z(0) = \hat{u}(k(0), \tau(0)) = z(s) e^{-\left[\left(-\bar{\kappa} k^2(-\frac{1}{2\gamma}) e^{-2\gamma s} + \gamma s\right) + \bar{\kappa} k^2(-\frac{1}{2\gamma})\right]}$$

$$= \hat{u}\left(e^{-\gamma s} k, s+t\right) e^{-\left(\gamma s + \frac{\bar{\kappa} k^2}{2\gamma}(e^{-2\gamma s} - 1)\right)} \qquad \text{Let } s = -\tau$$

$$= \hat{u}_0(e^{\gamma\tau} k) e^{-\left(-\gamma\tau + \frac{\bar{\kappa} k^2}{2\gamma}(e^{2\gamma\tau} - 1)\right)}$$

$$= \hat{u}_0(e^{\gamma\tau} k) e^{\gamma\tau + \frac{\bar{\kappa} k^2}{2\gamma}(1 - e^{2\gamma\tau})}$$

$$\hat{u}(k,\tau) = \left(\pi\,\delta(k e^{\gamma\tau}) - \frac{1}{ike^{\gamma\tau}}\right)$$

$$= e^{-\frac{\bar{\kappa}k^2}{2\gamma}(e^{2\gamma\tau}-1)}$$

$$u(z,\tau) = \int_{\mathbb{R}} e^{ikz}\left(\pi\,\delta(ke^{\gamma\tau}) + \frac{1}{ike^{\gamma\tau}}\right) e^{-\frac{\bar{\kappa}k^2}{2\gamma}(e^{2\gamma\tau}-1)}\, e^{\gamma\tau}\, dk$$

Let $k\tau = u \Rightarrow \tau\, dk = du$

$\to 1$ as $\tau \to \infty$

$$u(z,0) = H(z) = \frac{1}{\tau}\int_{\mathbb{R}} e^{\frac{iuz}{\tau}}\left(\pi\,\delta\!\left(\frac{u}{\tau}\right) + \frac{\tau}{iu}\right) e^{\frac{u^2}{2\gamma\tau}}\, e^{u\gamma}\, du$$

$$= \frac{1}{\tau}\int_{\mathbb{R}}\left(\pi + \frac{\tau}{iu}\right) e^{u\gamma}\, du$$

$$e^{-\frac{\bar{\kappa}k^2}{2\gamma}(e^{2\gamma\tau}-1)} = e^{-\frac{\bar{\kappa}k^2}{2\gamma}e^{2\gamma\tau}(1-e^{-2\gamma\tau})}$$ Let $ke^{\gamma\tau} = u$

$$u(z,0) = \int_{\mathbb{R}} e^{ie^{-\gamma\tau}u}\left(\pi\,\delta(u) + \frac{1}{iu}\right) e^{-\frac{\bar{\kappa}u^2}{2\gamma}(1-e^{-2\gamma\tau})}$$

$$= \int_{\mathbb{R}} e^{iu}\left(\pi\,\delta(u) + \frac{1}{iu}\right) e^{-\bar{\kappa}u^2/2\gamma}\, du$$

$$= \cancel{\int_{\mathbb{R}} e^{iuz}\,\pi\,\delta(u)\, e^{-\bar{\kappa}u^2/2\gamma}} + \int_{\mathbb{R}} e^{iuz}\cdot\frac{1}{iu}\, e^{-\bar{\kappa}u^2/2\gamma}$$

$$u(z,\tau) \sim \pi + \int_{\mathbb{R}} e^{iuz}\cdot\frac{1}{iu}\cdot e^{-\bar{\kappa}u^2/2\gamma}\, du$$ Q: Why is this $\mathbb{C}$?

$$\int_{\mathbb{R}} e^{iuz}\cdot\frac{1}{iu}\cdot e^{-\bar{\kappa}u^2/2\gamma}\, du = \int_0^{\infty} e^{iuz}\frac{1}{iu}\, e^{-\bar{\kappa}u^2/2\gamma}\, du + \int_{-\infty}^{0} e^{iuz}\cdot\frac{1}{iu}\cdot e^{-\bar{\kappa}u^2/2\gamma}\, du$$

$$= \int_0^{\infty ?} \frac{2\sin(uz)}{u}\, e^{-\bar{\kappa}u^2/2\gamma}\, du - \int_0^{\infty} \frac{e^{-iuz}}{iu}\, e^{-\frac{\bar{\kappa}u^2}{2\gamma}}\, du$$

$$\Rightarrow u(z,\tau) \sim \pi + 2\int_0^{\infty} \frac{\sin(uz)}{u}\, e^{-\frac{\bar{\kappa}u^2}{2\gamma}}\, du$$ (numerically compute?)

$$\Downarrow$$

$$u_z \sim 2\int_0^{\infty} \cos(uz)\, e^{-\bar{\kappa}u^2/2\gamma}\, du$$ as $\tau \to \infty$

7a) Let $\{f_n\}$ be a seq of funcs defined on Ω.

Ptws: We say $f_n \to f$ ptws if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ st $\forall n \geq N$ we get $|f_n(x_0) - f(x)| < \varepsilon$.

Here we only care about the fixed pt x_0 and N can change

Asymp: For a fixed $N \in \mathbb{N}$ we say $f_n \to f$ asymp. if $\forall \varepsilon > 0\ \exists \delta > 0$ st $|x - x_0| < \delta$ gives $|f_n(x) - f(x)| < \varepsilon$.

Here we only care about the neighborhood of x_0, not x_0 itself. and N is fixed

Ex: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

For any fixed $|x_0| < \infty$, $\sum_{n=0}^{\infty} \frac{x_0^n}{n!} = e^{x_0}$ so it conv ptws

$\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1} \xrightarrow[x\to\infty]{} \infty$ thus we don't have asymp conv

Ex: $\sum_{n=0}^{\infty} \frac{n!}{x^n}$

$\frac{(n+1)!}{x^{n+1}} \cdot \frac{x^n}{n!} = \frac{n+1}{x}$ $\xrightarrow[n\to\infty]{} \infty$ doesn't conv ptws for any fixed x_0

$\xrightarrow[x\to\infty]{} 0$ conv asymp

b) The series needs to be about a regular singular pt so that we have an asymp seq $\{(z-z_0)^j\}_{j=n}^{\infty}$ where n is a fixed integer. If the pt is irregular, the series looks like $\sum_{j=-\infty}^{\infty} a_j (z-z_0)^j$, so wouldn't have a locaing term.